Study Questions Homework 3 Foundations of Computational Math 1 Fall 2020

Problem 3.1

Let $f(x)$ be a smooth function and let $p_n(x)$ be a polynomial of degree n that satisfies the Hermite-Brikhoff interpolation conditions for the point x_0

$$
\begin{aligned}
p_n(x_0) &= f(x_0)\\
p_n'(x_0) &= f'(x_0)\\
p_n''(x_0) &= f''(x_0)\\
&\vdots\\
p_n^{(n)}(x_0) &= f^{(n)}(x_0).
\end{aligned}
$$

3.1.a. Construct the Newton form of $p_n(x)$ using the Newton divided difference table. Identify and explain any structure in the divided difference table.

3.1.b. Using the basis that arises from the Newton form of $p_n(x)$, derive linear equations that impose the Hermite-Birkhoff interpolation conditions and therefore define the divided differences.

3.1.c. Show that $p_n(x)$ is unique and that the coefficients determined by solving the linear system are the same as those determined by using the divided difference table.

Problem 3.2

(3.2.a) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist:

$$
\begin{aligned}
f(0) = 0, &\quad f'(0) = 1\\
f(1) = 3, &\quad f'(1) = 6
\end{aligned}
$$

(3.2.b) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist:

$$
\begin{aligned}
f(0) = 0, &\quad f'(0) = 0\\
f(1) = 3, &\quad f'(1) = 6\\
f(2) &= 1
\end{aligned}
$$

(3.2.c) Determine the polynomial of minimal degree that matches the following conditions on f or show that it does not exist. (Note that this is not an Hermite-Birkhoff form of interpolation problem.)

$$f(0) = 3$$
$$f'(0) = 5, \quad f'(1) = 10, \quad f'(2) = 10$$

Problem 3.3

Text exercise 8.10.9 on page 377

Problem 3.4

Let $f(x) = \cos 8x$ on $0 \le x \le 1$. Suppose $f(x)$ is to be approximated by a piecewise linear interpolating function, $g_1(x)$. The accuracy required is

$$\forall 0 \le x \le 1, \quad |f(x) - g_1(x)| \le 10^{-6}$$

Determine a bound on $h = x_i - x_{i-1}$ for uniformly spaced points that satisfies the required accuracy.

Problem 3.5

Suppose we want to approximate a function$f(x)$ on the interval $[a, b]$ with a piecewise quadratic interpolating polynomial, $g_2(x)$, with a constant spacing, h, of the interpolation points $a = x_0 < x_1 \ldots < x_n = b$. That is, for any $a \le x \le b$, the value of $f(x)$ is approximated by evaluating the quadratic polynomial that interpolates f at x_{i-1}, x_i, and x_{i+1} for some i with $x = x_i + sh$, $x_{i-1} = x_i - h$, $x_{i+1} = x_i + h$ and $-1 \le s \le 1$. (How i is chosen given a particular value of x is not important for this problem. All that is needed is the condition $x_{i-1} \le x \le x_{i+1}$.)

Suppose we want to guarantee that the **relative error** of the approximation is less than 10^{-d}, i.e., d digits of accuracy. Specifically,

$$\frac{|f(x) - g_2(x)|}{|f(x)|} \le 10^{-d}.$$

(It is assumed that $|f(x)|$ is sufficiently far from 0 on the interval $[a, b]$ for relative accuracy to be a useful value.) Derive a bound on h that guarantees the desired accuracy and apply it to interpolating $f(x) = e^x \sin x$ on the interval $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$ with relative accuracy of 10^{-4}. (The sin is bounded away from 0 on this interval.)

Compare your predicted accuracy to the accuracy you achieve by forming $g_2(x)$ for h's that satisfy your bound and h's that do not.

Problem 3.6

Consider the following data

$$(x_0, f_0) = (1, 0), \quad (x_1, f_1) = (2, 2),$$
$$(x_2, f_2) = (4, 12), \quad (x_3, f_3) = (5, 21)$$

3.6.a. Determine the quadratic interpolating polynomial, $p_2(x)$, for points (x_0, f_0), (x_1, f_1), (x_2, f_2). Estimate $f(3)$ using $p_2(x)$.

3.6.b. Determine the quadratic interpolating polynomial, $\tilde{p}_2(x)$, for points (x_1, f_1), (x_2, f_2), (x_3, f_3). Estimate $f(3)$ using $\tilde{p}_2(x)$.

3.6.c. Estimate $f(3)$ using a cubic interpolating polynomial $p_3(x)$.

3.6.d. Estimate the errors $|f(3) - p_2(x)|$ and $|f(3) - \tilde{p}_2(x)|$ an use the estimates to determine a range of values in which you expect $f(3)$ to reside. How does the value of $p_3(3)$ relate to this interval?

3.6.e. Write the piecewise linear interpolant $g_1(x)$ that uses all of the data points in the form that specifies the set of intervals and the linear polynomial on each interval. Estimate $f(3)$ using $g_1(x)$.

3.6.f. Determine the cardinal basis form of $g_1(x)$. Verify that your cardinal basis form satisfies the interpolation constraints.