Study Questions Homework 6 Foundations of Computational Math 1 Fall 2020

Problem 6.1

Consider the following linear multistep method:

$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$

6.1.a. Determine, p, the order of consistency of the method.

6.1.b. Determine the coefficient, C_{p+1}, in the discretization error d_n.

6.1.c. Consider the application of the method to $y' = 0$ with $y_0 = 0$ and $y_1 = \epsilon$, i.e., a perturbed initial condition. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Problem 6.2

Consider the following linear multistep method:

$$y_n = y_{n-2} + \frac{h}{3}(f_n + 4f_{n-1} + f_{n-2})$$

The method is 0-stable but it is weakly stable.

6.2.a. Determine the discretization error d_n.

6.2.b. Consider the application of the method to $y' = \lambda y$. Write the recurrence that yields y_n.

6.2.c. Let y_n, $n = 0, 1, \ldots$ be the numerical solution of $y' = \lambda y$ from the previous part of the problem. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Problem 6.3

Adapt the techniques used to derive the Adams Moulton 2-step method with constant step

$$y_n = y_{n-1} + h\left(\frac{5}{12}f_n + \frac{8}{12}f_{n-1} - \frac{1}{12}f_{n-2}\right).$$

to find the expression for a nonconstant stepsize 2-step Adams Moulton method with stepsizes $h_n = t_n - t_{n-1}$ and $h_{n-1} = t_{n-1} - t_{n-2}$. Give the result in the form:

$$y_n = y_{n-1} + h\left(\beta_0(h_n, h_{n-1})f_n + \beta_1(h_n, h_{n-1})f_{n-1} - \beta_2(h_n, h_{n-1})f_{n-2}\right).$$

where the real coefficients $\beta_i(h_n, h_{n-1})$ are functions of the two stepsizes h_n and h_{n-1}.

Problem 6.4

Recall, we have examined the polynomials $\rho(\xi)$ and $\sigma(\xi)$ associated with a linear multistep method. $\rho(\xi)$ is related to the analysis of strong, weak and 0-stability of the method and $\mu(\xi) = \rho(\xi) - h\lambda\sigma(\xi)$ is used to determine the absolute stabililty properties and region of the method. All three parts of this question relate in some way to these three polynomials.

6.4.a

i. What stability properties of the method can be examined by looking at the roots of $\sigma(\xi)$? (Take care when $\sigma(\xi)$ has lower degree than $\rho(\xi)$.)

ii. Explain the statement "The Adams methods are as strongly stable as any linear multistep method can possibly be."

iii. The motivation for the design of the BDF methods is stiff decay. Explain how the form of the BDFs is linked to this motivation.

6.4.b

Consider the linear multistep method:

$$y_n - y_{n-2} = 2hf_{n-1}$$

Discuss the absolute stability properties of the method. You may do this by determining the boundary of the absolute stability region or by other means.

6.4.c

Consider the linear multistep method:

$$y_n = y_{n-1} + h\left(\frac{9}{16}f_n + \frac{6}{16}f_{n-1} + \frac{1}{16}f_{n-2}\right)$$

(i) Is the method convergent?

(ii) The method is **not** an Adams Moulton method. Examine the absolute stability properties of this method and identify the main advantage or disadvantages this method compared to the 2-step Adams Moulton method. You do not have to determine the entire boundary to solve this problem.

Problem 6.5

Assume you have an implicit k step linear multistep method of the form

$$y_n = h\beta_0 f_n + \sum_{i=1}^{k} (h\beta_i f_{n-i} - \alpha_i y_{n-i}) = h\beta_0 f_n + S_*$$

that has order p, i.e.,

$$y(t_n) = h\beta_0 f(y(t_n)) + \sum_{i=1}^{k} (h\beta_i f(y(t_{n-i})) - \alpha_i y(t_{n-i})) + O(h^{p+1})$$

where the t argument to f has been suppressed for convenience.

Suppose you apply a $P(EC)^m E$ method to solve approximately this implicit equation to determine y_n and the predictor is assumed to have order $\ell < p$ accuracy, i.e.,

$$y(t_n) = y_n^{[0]} + O(h^{\ell+1})$$

$$y_n^{[j]} = h\beta_0 f(y_n^{[j-1]}) + S_*$$

(6.5.a) Assume that $y_{n-i} = y(t_{n-i})$ for $i = 1, \ldots, k$ and show that each iteration of the EC step increases the order of accuracy of $y_n^{[j]}$ by 1, i.e.,

$$y(t_n) = y_n^{[j]} + O(h^{l+1+j}) + O(h^{p+1})$$

(6.5.b) What order ℓ for the predictor would you recommend be used in practice and why?

Problem 6.6

6.6.a

Solutions to ODE initial value problems often satisfy invariants, i.e., a condition on $y(t)$ that is true for all t in the interval defined by the problem. For example, the solution $y(t) \in \mathbb{R}^m$ to

$$y' = f(y,t), \quad y(t_0) = y_0$$

where $f(y,t) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is Lipchitz in y could have a constant size as measured by the vector 2-norm in $\mathbb{R}^m$, i.e.,

$$\|y_0\|_2^2 = y_0^T y_0 = \|y(t)\|_2^2 = y(t)^T y(t)$$

for all t in the interval of the problem.

What condition must hold for $y(t)$ and $f(y,t)$ to give a solution that is invariant in the vector 2-norm in $\mathbb{R}^m$? Justify your answer.

6.6.b

If a solution to an IVP satisfies an invariant it is of interest to know which numerical methods preserve that invariant in the numerical solution (assuming exact arithmetic, i.e., no roundoff).

The system of two ODEs

$$Y'(t) = MY(t), \quad Y(0) = Y_0$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\omega > 0$ has a solution for which $\|Y(t)\|_2^2 = \|Y_0\|_2^2$ where $Y(t) \in \mathbb{R}^2$ contains $y_1(t)$ and $y_2(t)$ as its components and $M \in \mathbb{R}^{2\times 2}$. In fact, for any $Y_0 \in \mathbb{R}^2$ the solution stays on the circle with radius $\|Y_0\|_2^2$.

Derive expressions for the numerical solution Y_n and $\|Y_n\|_2^2$ that results from applying the Trapezoidal Rule to the problem in part above and use them to determine if the Trapezoidal Rule preserves $\|Y_n\|_2^2 = \|Y_0\|_2^2$?

Problem 6.7

Consider the Runge Kutta method called the implicit midpoint rule given by:

$$\begin{aligned} \hat{y}_1 &= y_{n-1} + \frac{h}{2} f_1 \\ f_1 &= f(t_{n-1} + \frac{h}{2}, \hat{y}_1) \\ y_n &= y_{n-1} + h f_1 \end{aligned}$$

An alternate form of the the method is given by:

$$y_n = y_{n-1} + hf\big(\frac{t_n + t_{n-1}}{2}, \frac{y_n + y_{n-1}}{2}\big)$$

Show that the two forms are identical.

Problem 6.8

Consider the general form of a 2-stage Explicit RK method:

$$\hat{y}_1 = y_{n-1}, \quad f_1 = f(t_{n-1}, \hat{y}_1)$$

$$\hat{y}_2 = y_{n-1} + \alpha_{21} f_1, \quad f_2 = f(t_{n-1} + \gamma_2 h, \hat{y}_2)$$

$$y_n = y_{n-1} + h\left(\beta_1 f_1 + \beta_2 f_2\right)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_2 & \alpha_{21} & 0 \\ \hline & \beta_1 & \beta_2 \end{array}$$

$$\gamma_2 = \alpha_{21}$$

6.8.a. Determine the set of equations that the free parameters must satisfy in order to achieve method with order 2.

6.8.b. Is there a single such method? If so prove it. If not discuss the number of free parameters and give examples of methods and potential parameterized tables that define families of methods.