Study Questions Homework 2 Foundations of Computational Math 1 Fall 2021

Problem 2.1

Recall that a unit lower triangular matrix $L \in \mathbb{R}^{n\times n}$ is a lower triangular matrix with diagonal elements $e_i^T L e_i = \lambda_{ii} = 1$. An elementary unit lower triangular column form matrix, L_i, is an elementary unit lower triangular matrix in which all of the nonzero subdiagonal elements are contained in a single column. For example, for $n = 4$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ \lambda_{31} & 0 & 1 & 0 \\ \lambda_{41} & 0 & 0 & 1 \end{pmatrix} \quad L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda_{32} & 1 & 0 \\ 0 & \lambda_{42} & 0 & 1 \end{pmatrix} \quad L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda_{43} & 1 \end{pmatrix}$$

2.1.a. Show that any elementary unit lower triangular column form matrix , $L_i \in \mathbb{R}^{n\times n}$, can be written as the identity matrix plus an outer product of two vectors, i.e., $L_i = I + v_i w_i^T$ where $v_i \in \mathbb{R}^n$ and $w_i \in \mathbb{R}^n$. (This is often called a rank-1 update of a matrix.) Make sure the structure required in v_i and w_i is clearly stated.

2.1.b. Show that L_i has an inverse and it is an elementary unit lower triangular column form matrix.

2.1.c. Consider the matrix vector product $y = L_i x$ where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, and $L_i \in \mathbb{R}^{n\times n}$ is an elementary unit lower triangular column form matrix. Determine an efficient algorithm to compute the product and its computational/storage complexity.

2.1.d. Suppose $L_i \in \mathbb{R}^{n\times n}$ and $L_j \in \mathbb{R}^{n\times n}$ are elementary unit lower triangular column form matrices with $1 \leq i < j \leq n-1$. Consider the matrix product $B = L_i L_j$. Determine an efficient algorithm to compute the product and its computational/storage complexity.

2.1.e. Suppose $L_i \in \mathbb{R}^{n\times n}$ and $L_j \in \mathbb{R}^{n\times n}$ are elementary unit lower triangular column form matrices with $1 \leq j \leq i \leq n-1$. Consider the matrix product $B = L_i L_j$. Determine an efficient algorithm to compute the product and its computational/storage complexity.

2.1.f. Let $L \in \mathbb{R}^{n\times n}$ be a unit lower triangular matrix. Show that $L = L_1 L_2 \cdots L_{n-1}$ where L_i is an elementary unit lower triangular column form matrix for $1 \leq i \leq n-1$.

2.1.g. Express the column-oriented algorithm for solving $Lx = b$ where L is a unit lower triangular matrix in terms of operations involving unit lower triangular column form matrices.

Problem 2.2

2.2.a

An **elementary unit upper triangular column form matrix** $U_i \in \mathbb{R}^{n\times n}$ is of the form

$$I + u_i e_i^T$$

where $u_i^T e_j = 0$ for $i \leq j \leq n$. This matrix has 1 on the diagonal and the nonzero elements of u_i appear in the i-th column above the diagonal.

For example, if $n = 3$ then

$$\begin{aligned} U_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \mu_{13} \\ \mu_{23} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \mu_{13} \\ 0 & 1 & \mu_{23} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $U \in \mathbb{R}^{n\times n}$ be a unit upper triangular matrix. Show that the factorization

$$U = U_n U_{n-1} \cdots U_2,$$

where $U_i = I + u_i e_i^T$ and the nonzeros of u_i are the nonzeros in the i-th column of U above the diagonal, can be formed without any computations.

2.2.b

Now suppose that $U \in \mathbb{R}^{n\times n}$ is a upper triangular with diagonal elements μ_{ii}. Let $S_i \in \mathbb{R}^{n\times n}$ be a diagonal matrix with its i-th diagonal element $e_i^T S_i e_i = \mu_{ii}$ and all of the other diagonal elements $e_j^T S_i e_j = 1$ for $i \neq j$.

For example, if $n = 3$ then

$$\begin{aligned} S_1 &= \begin{pmatrix} \mu_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S_2 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix} \end{aligned}$$

Let $U_i = I + u_i e_i^T$ and the nonzeros of u_i be the nonzeros in the i-th column of U above the diagonal. (This implies that $U_1 = I$)

Show that

$$U = (S_n U_n)(S_{n-1} U_{n-1}) \cdots (S_2 U_2)(S_1 U_1).$$

Note that U may be singular so some μ_{ii} may be 0. Therefore, a proof based on expressing the algorithm for the solution of $Ux = b$ in terms of U_i^{-1} and S_i^{-1}, as is done in the next part of the question, is not applicable.

2.2.c

From the factorization of the previous part of the problem, derive an algorithm to solve $Ux = b$ given U is an $n \times n$ nonsingular upper triangular matrix. Describe the basic computational primitives required.

Problem 2.3

Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular and that $A = LU$ is its LU factorization. Give an algorithm that can compute, $e_i^T A^{-1} e_j$, i.e., the (i, j) element of A^{-1} in approximately $(n - j)^2 + (n - i)^2$ operations.

Problem 2.4

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;
- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.4.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.4.b. What is the pattern of element values in L and U for any n?

Problem 2.5

Suppose you have the LU factorization of an $i \times i$ matrix $A_i = L_i U_i$ and suppose the matrix A_{i+1} is an $i+1 \times i+1$ matrix formed by adding a row and column to A_i, i.e.,

$$A_{i+1} = \begin{pmatrix} A_i & a_{i+1} \\ b_{i+1}^T & \alpha_{i+1,i+1} \end{pmatrix}$$

where a_{i+1} and b_{i+1} are vectors in $\mathbb{R}^i$ and $\alpha_{i+1,i+1}$ is a scalar.

2.5.a. Derive an algorithm that, given L_i, U_i and the new row and column information, computes the LU factorization of A_{i+1} **and identify the conditions under which the step will fail**.

2.5.b. What computational primitives are involved?

2.5.c. Show how this basic step could be used to form an algorithm that computes the LU factorization of an $n \times n$ matrix A.

Problem 2.6

Consider a symmetric matrix A, i.e., $A = A^T$.

2.6.a. Consider the use of Gauss transforms to factor $A = LU$ where L is unit lower triangular and U is upper triangular. **You may assume that the factorization does not fail.** Show that $A = LDL^T$ where L is unit lower triangular and D is a matrix with nonzeros on the main diagonal. i.e., elements in positions (i, i), and zero everywhere else, by demonstrating that L and D can be computed by applying Gauss transforms appropriately to the matrix A.

2.6.b. For an arbitrary symmetric matrix the LDL^T factorization will not always exist due to the possibility of 0 in the (i, i) position of the transformed matrix that defines the i-th Gauss transform. Suppose, however, that A is a **positive definite** symmetric matrix, i.e., $x^T Ax > 0$ for any vector $x \neq 0$. Show that the diagonal element of the transformed matrix A that is used to define the vector l_i that determines the Gauss transform on step i, $M_i^{-1} = I - l_i e_i^T$, is always positive and therefore the factorization will not fail. Combine this with the existence of the LDL^T factorization to show that, in this case, the nonzero elements of D are in fact positive.

Problem 2.7

Suppose you are computing a factorization of the $A \in \mathbb{C}^{n\times n}$ with partial pivoting and at the beginning of step i of the algorithm you encounter the the transformed matrix with the form

$$TA = A^{(i-1)} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & A_{i-1} \end{pmatrix}$$

where $U_{11} \in \mathbb{R}^{i-1\times i-1}$ and nonsingular, and $U_{12} \in \mathbb{R}^{i-1\times n-i+1}$ contain the first $i-1$ rows of U. Show that if the first column of A_{i-1} is all 0 then A must be a singular matrix.

Problem 2.8

Suppose $A \in \mathbb{R}^{n\times n}$ is a nonsymmetric nonsingular diagonally dominant matrix with the following nonzero pattern (shown for $n = 6$)

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

It is known that a diagonally dominant (row or column dominant) matrix has an LU factorization and that pivoting is not required for numerical reliability.

2.8.a. Describe an algorithm that solves $Ax = b$ as efficiently as possible.

2.8.b. Given that the number of operations in the algorithm is of the form $Cn^k + O(n^{k-1})$, where C is a constant independent of n and $k > 0$, what are C and k?