Homework 6 Foundations of Computational Math 1 Fall 2021

These study questions relate to the material on solving scalar and systems of nonlinear equations.

Problem 6.1

Let $\phi(x) : [a, b] \to [a, b]$ be a continuous function. Show that if $\phi(x)$ is a contraction mapping on $[a, b]$ then the sequence $\{x^{(k)}\}$ defined by $x^{(k+1)} = \phi(x^{(k)})$ is a Cauchy sequence.

Problem 6.2

Let $f(x) = x^3 - 3x + 1$. This polynomial has three distinct roots.

(6.2.a) Consider using the iteration function

$$\phi_1(x) = \frac{1}{3}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_1(x)$ and how would you choose $x^{(0)}$ for each computable root?

(6.2.b) Consider using the iteration function

$$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_2(x)$ and how would you choose $x^{(0)}$ for each computable root?

(6.2.c) For each of the roots you identified as computable using either $\phi_1(x)$ or $\phi_2(x)$, apply the iteration to find the values of the roots. (You need not turn in any code, but using a simple program to do this is recommended.)

Problem 6.3

Textbook, p. 283, Problem 2

Problem 6.4

Textbook, p. 283, Problem 6

Problem 6.5

Textbook, p. 284, Problem 8

Problem 6.6

6.6.a

Consider the polynomial

$$p(x) = x^3 - x - 5$$

The following three iterations can be derived from $p(x)$.

$$\phi_1(x) = x^3 - 5$$
$$\phi_2(x) = \sqrt[3]{x + 5}$$
$$\phi_3(x) = 5/(x^2 - 1)$$

The polynomial $p(x)$ has a root $r > 1.5$. Determine which of the $\phi_i(x)$ produce an iteration that converges to r.

6.6.b

Consider the function

$$f(x) = xe^{-x} - 0.06064$$

i. Write the update formula for Netwon's method to find the root of $f(x)$.

ii. $f(x)$ has a root of $\alpha = 0.06469263\ldots$. If you run Newton's method with $x_0 = 0$ convergence occurs very quickly, e.g., in double precision $|f(x_4)| \approx 10^{-10}$. However, if x_0 is large and negative or if x_0 is near 1 convergence is much slower until there is a very rapid improvement in accuracy in the last one or two steps. For example, if $x_0 = 0.98$ then

$$|f(x_{47})| \approx 0.6 \times 10^{-2}$$
$$|f(x_{49})| \approx 1.5 \times 10^{-5}$$
$$|f(x_{50})| \approx 2.8 \times 10^{-10}$$

Explain this behavior. You might find it useful to plot $f(x)$ and run a few examples of Newton's method.

Problem 6.7

Consider the fixed point iteration by the function

$$\phi(x) = x - \frac{(x^2 - 3)}{(x^2 + 2x - 3)}$$

The value $\alpha = \sqrt{3}$ is a fixed point for this iteration. Provide justification to all of your answers for the following:

(6.7.a) Show that there exists a nontrivial interval $\alpha - \delta < x < \alpha + \delta$ with $\delta > 0$ such that the iteration defined by $\phi(x)$ converges to α for any x_0 in the interval.

(6.7.b) Is the order of convergence on this interval linear ($p = 1$) or higher ($p \geq 2$)?

(6.7.c) Show that for $x > \alpha$ we have

$$\alpha < \phi(x) < x$$

and use this to explain the convergence of the iteration when $x_0 > \alpha$.

(6.7.d) Plot or otherwise enumerate values of the curves $y = \phi(x)$ and $y = x$ on the interval $0 < x < \beta$ for $\beta > \alpha$. Use the information to examine the behavior of the iteration for $0 < x_0 < \alpha$. For what subinterval, if any, do you expect convergence to α?

Problem 6.8

Textbook, page 330, Problem 6

Problem 6.9

Consider the generic cubic polynomial with three distinct roots 0, $\rho > 0$ and $-\rho$ with the following form and properties:

$$f(x) = x(x - \rho)(x + \rho) = x^3 - \rho^2 x, \quad \rho > 0$$

$$f'(x) = 3x^2 - \rho^2 = 3(x^2 - \alpha^2) = 3(x - \alpha)(x + \alpha), \quad \alpha = \rho/\sqrt{3} > 0$$

- f is concave on $-\infty < x < -\alpha$.
- f is convex on $\alpha < x < \infty$

- f has the standard cubic form on $-\alpha < x < \alpha$

$$f(x) > 0 \quad \text{on} \quad -\alpha < x < 0$$

$$f(0) = 0$$

$$f(x) < 0 \quad \text{on} \quad 0 < x < \alpha$$

6.9.a Write the iteration $\phi(x)$ that defines Newton's method. For this problem, you may find it useful to rewrite the standard form into

$$\phi(x) = \gamma(x) \;\; x.$$

6.9.b Determine the point $0 < \xi < \alpha$ such that if $x_0 = \xi$ Newton's method cycles between two point (in exact arithmetic).

6.9.c Using the facts given above and any others you prove, describe the behavior of Newton's method on the following intervals and justify your claims:

$$x > \rho$$

$$\alpha < x < \rho$$

$$-\xi < x < \xi$$

Problem 6.10

A cubic polynomial with a simple root at -1 and a double root at 2 is given by

$$f(x) = (x+1)(x-2)^2 = x^3 - 3x^2 + 4.$$

Consider the following iteration function

$$\phi(x) = \frac{2x^3 - 3x^2 - 4}{3x^2 - 6x} = \frac{2x^2 + x + 2}{3x}$$

that, given $x^{(0)}$, defines the iteration

$$x^{(k+1)} = \phi(x^{(k)}).$$

We therefore have

$$\phi'(x) = \frac{2(x^2-1)}{3x^2}.$$

All answers must have sufficient justification. You might find it useful to sketch $f(x)$.

6.10.a Suppose $x_{\pm}^{(0)} = \pm\epsilon$, with $\epsilon > 0$ a small number, are two possible initial iterates very close to 0. Where are the two points $x_{\pm}^{(1)}$? (Ignore any numerical floating point issues and assume this is in exact arithmetic, i.e., real arithmetic.)

6.10.b Suppose the initial iterate satisfies $x^{(0)} > 2$. Does the sequence $x^{(k)}$ converge to the root 2?

6.10.c Suppose the initial iterate satisfies $x^{(0)} < -1$. Does the sequence $x^{(k)}$ converge to the root -1?

6.10.d Are there any $x^{(0)}$ values for which $x^{(1)}$ is not defined?

6.10.e Suppose the initial iterate, $x^{(0)}$, is such that the sequence $x^{(k)}$ converges to the root 2. What is the convergence rate?

6.10.f Suppose the initial iterate, $x^{(0)}$, is such that the sequence $x^{(k)}$ converges to the root -1. What is the convergence rate?

6.10.g Given your results for the previous questions, summarize the behavior of the iteration $x^{(k+1)} = \phi(x^{(k)})$ for any initial iterate, $x^{(0)} \in \mathbb{R}$.

Problem 6.11

Consider the polynomial

$$p(x) = x^3 - 3x - 3$$

and the following iterations

$$\phi_1(x) = \sqrt[3]{3x+3}$$

$$\phi_2(x) = \frac{x^3}{3} - 1$$

$$\phi_3(x) = \frac{2x^3+3}{3x^2-3}$$

6.11.a Determine which of the $\phi_i(x)$ have all of the roots of $p(x)$ as fixed points.

6.11.b $p(x)$ has a root $r > 1$. Determine which $\phi_i(x)$ converge to r for some nontrivial interval and the rates of convergence for the $\phi_i(x)$ that converge to r?

Problem 6.12

6.12.a

Consider the system of equations in $\mathbb{R}^2$

$$\xi^2 + \eta^2 = 4$$
$$e^\xi + \eta = 1$$

Show that the system has two solutions in $\mathbb{R}^2$, one with $\xi > 0$ and $\eta < 0$ and one with $\xi < 0$ and $\eta > 0$.

6.12.b

Consider the following function and iteration:

$$G_1(x) = \begin{pmatrix} \gamma_1(\xi,\eta) \\ \gamma_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \ln(1-\eta) \\ -\sqrt{4-\xi^2} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k,\eta_k) \\ \gamma_2(\xi_k,\eta_k) \end{pmatrix} = \begin{pmatrix} \ln(1-\eta_k) \\ -\sqrt{4-\xi_k^2} \end{pmatrix}$$

(i) Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.

(ii) Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.

(iii) Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.

(iv) Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.

6.12.c

Consider the following function and iteration:

$$G_2(x) = \begin{pmatrix} \gamma_1(\xi,\eta) \\ \gamma_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta^2} \\ 1-e^{\xi} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k,\eta_k) \\ \gamma_2(\xi_k,\eta_k) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta_k^2} \\ 1-e^{\xi_k} \end{pmatrix}$$

(i) Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.

(ii) Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.

(iii) Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.

(iv) Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.

Problem 6.13

Systems of Equations

Consider the following systems of equations in $\mathbb{R}^2$:

System 1:

$$\begin{aligned} \xi^2 + \eta^2 &= 4 \\ e^{\xi} + \eta &= 1 \end{aligned}$$

It was shown in the homework that the system has two solutions in $\mathbb{R}^2$, one with $\xi > 0$ and $\eta < 0$ and one with $\xi < 0$ and $\eta > 0$.

System 2:

$$F(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sin(\xi_1\xi_2) - \frac{\xi_2}{4\pi} - \frac{\xi_1}{2} \\ (1 - \frac{1}{4\pi})(e^{2\xi_1} - e) + \frac{e\xi_2}{\pi} - 2e\xi_1 \end{pmatrix}$$

$$J_F = \begin{pmatrix} 0.5(\xi_2\cos(\xi_1\xi_2) - 1) & 0.5\xi_1\cos(\xi_1\xi_2) - 1/(4\pi) \\ (2 - (1/(2\pi)))e^{2\xi_1} - 2e & e/\pi \end{pmatrix}$$

This system was used in the notes for nonlinear relaxation and Newton's method.

System 3:

$$F(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (\xi_1 + 3)(\xi_2^3 - 7) + 18 \\ \sin(\xi_2 e^{\xi_1} - 1) \end{pmatrix} \rightarrow x^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J_F = \begin{pmatrix} (\xi_2^3 - 7) & 3\xi_2^2(\xi_1 + 3) \\ \xi_2 e^{\xi_1} \cos(\xi_2 e^{\xi_1} - 1) & e^{\xi_1} \cos(\xi_2 e^{\xi_1} - 1) \end{pmatrix}$$

This system was used in the notes for nonlinear relaxation, Newton's method and Broyden's method.

Algorithm Implementation Tasks

(6.13.a) Implement Newton's method to solve a system of two nonlinear equations in a generic fashion, i.e., the code should accept a user-supplied subroutine for $F(x)$ and a user-supplied subroutine to compute the Jacobian $J_F(x)$. The termination criterion should be controlled by a user-set parameters.

(6.13.b) Implement Broyden's method to solve a system of two nonlinear equations in a generic fashion, i.e., the code should accept user-supplied subroutins for $F(x)$. You may use a user-set constant $0 < \alpha_k \leq 1$ as the stepsize. The termination criterion should be controlled by a user-set parameters such as tolerances or maximum number of iterations or both.

(6.13.c) Implement a Nonlinear Jacobi-Newton (one-step) to solve a system of two nonlinear equations in a generic fashion, i.e., the code should accept a user-supplied subroutine for $F(x)$ and a user-supplied subroutine to compute the appropriate elements of the Jacobian $J_F(x)$. The termination criterion should be controlled by a user-set parameters such as tolerances or maximum number of iterations or both.

(6.13.d) Implement a Nonlinear Gauss-Seidel-Newton (one-step) to solve a system of two nonlinear equations in a generic fashion, i.e., the code should accept a user-supplied subroutine for $F(x)$ and a user-supplied subroutine to compute the appropriate elements of the Jacobian $J_F(x)$. The termination criterion should be controlled by a user-set parameters such as tolerances or maximum number of iterations or both.

Empirical Tasks

For each of the systems of equations and each of the methods:

(6.13.a) Consider the performance for a variety of initial conditions to solve the system. Attempt to identify a region in the plane for convergence to each of the roots you find.

(6.13.b) Compare to the work required to achieve a given accuracy of residual, $\|F(x)\|$ or error $\|x - x^*\|$ starting all of the methods at the same initial condition assuming there is an intersection between the domains of convergence for each root you select for consideration.

(6.13.c) For the nonlinear relaxation Gauss-Seidel schedule you may use either or both of the orderings for the equation/unknown indices. Make sure you clearly state those used.

(6.13.d) Discuss and support with empirical evidence how your observed behaviors compare to those seen in the notes and the behavior predicted by the theory.