Written Problems Assignment 3 Foundations of Computational Math 1 Fall 2024

The solutions are due on Canvas by 11:59 PM on Tuesday November 26, 2024

Problem 3.1

Suppose x and y are two sparse vectors stored with their elements and indices in a compresed format that assumes the elements are stored in increasing order of their indices. Describe an algorithm to evaluate

$$z \leftarrow x + y$$

that does not make use of scatter/gather as in the notes. Compare the complexity of the two approaches.

Problem 3.2

Consider the block tridiagonal matrix associated with an $n \times n$ grid discretization of the partial differential $u_{\xi,\xi} + u_{\eta,\eta} = g$ on a two-dimensional domain.

The matrix is $n^2 \times n^2 = N \times N$ where $N = n^2$, with $n \times n$ blocks $T_i \in \mathbb{R}^{n \times n}$ $1 \leq i \leq n$ $E_i = -I_n \in \mathbb{R}^{n \times n}$ $1 \leq i \leq n$ and block tridiagonal structure given by

$$A = \begin{pmatrix} T_1 & E_1 & 0 & \cdots & \cdots & \cdots & 0 \\ E_2 & T_2 & E_2 & 0 & & & \vdots \\ 0 & E_3 & T_3 & E_3 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E_{n-1} & T_{n-2} & E_{n-2} & 0 \\ 0 & & \cdots & 0 & E_{n-1} & T_{n-1} & E_{n-1} \\ 0 & & & \cdots & 0 & E_n & T_n \end{pmatrix}$$

where T_i are tridiagonal and E_i are diagonal and dimensions $n \times n$

$$T_i = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 4 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & -1 & 4 & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & 4 & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

$$E_i = -I_n = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 0 & -1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & -1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Note: Be careful with N vs. n in your answer to this problem.

Suppose the system $Ax = b$ with $A \in \mathbb{R}^{N \times N}$ as above, $b \in \mathbb{R}^N$ and $x \in \mathbb{R}^N$ can be solved with both Jacobi and (forward) Gauss-Seidel. Assume that both of these methods are implemented using the preconditioner/residual form, i.e.,

$$x_{k+1} = x_k + P^{-1} r_k$$

is computed

$$\begin{aligned} \text{solve} \ \ P z_k &= r_k \\ x_{k+1} &= x_k + z_k \\ r_{k+1} &= r_k - A z_k \end{aligned}$$

with the appropriate P for each method. You may ignore any computations that are used to implement the termination check, i.e., determining that the iteration has reached its desired accuracy.

3.2.a. Determine the computational complexity of one step of Jacobi for solving $Ax = b$.

3.2.b. Determine the computational complexity of one step of Gauss-Seidel for solving $Ax = b$.

3.2.c. Assume both Jacobi and Gauss-Seidel start with the same x_0. How much faster in terms of the number of iterations must the method that takes more operations per step converge in order to take less total work than the method that takes fewer operations per step?

Problem 3.3

If $P(\xi)$ is a polynomial of degree d on $\mathbb{R}$ given by

$$P(\xi) = \gamma_0 + \gamma_1\xi + \cdots + \gamma_d\xi^d$$

then the matrix polynomial defined by P evaluated at $A \in \mathbb{R}^{n\times n}$ is the matrix

$$P(A) = \gamma_0 I + \gamma_1 A + \cdots + \gamma_d A^d \in \mathbb{R}^{n\times n}$$

(3.3.a) Suppose A is nonsingular and has n linearly independent eigenvectors. What are the eigenvalues and eigenvectors of the matrix $P(A)$?

(3.3.b) Consider a nonstationary method like Steepest Descent with the iteration defined by

$$x_k = x_{k-1} + \alpha_{k-1} r_{k-1}.$$

Can the idea of a matrix polynomial be used to relate r_k to r_{k-1} and to r_0? If so give the form of the polynomial and state the relationship. If not explain why not.