Graded Homework 1 Foundations of Computational Math 2 Spring 2021

The solutions are due by 11:59PM on January 27, 2021

Programming Exercise

The Methods Consider the following explicit Runge Kutta methods for numerically solving an ODE IVP:

- Forward Euler:

$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$$

- A one-parameter $s-2$ stage family with order $p=2$ (Slide Set 30 of FCM 1 Fall 20 on slide 18)

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \mu & \mu & 0 \\ \hline & 1-\frac{1}{2\mu} & \frac{1}{2\mu} \end{array}$$

$$\hat{y}_1 = y_{n-1} \rightarrow \hat{f}_1 = f_{n-1}$$

$$\hat{y}_2 = y_{n-1} + h\left(\mu \hat{f}_1\right) \rightarrow \hat{f}_2 = f(t_{n-1} + \mu h, \hat{y}_2)$$

$$y_n = y_{n-1} + h\left\{\left(1 - \frac{1}{2\mu}\right)\hat{f}_1 + \frac{1}{2\mu}\hat{f}_2\right\}$$

$\mu = 1$ is explicit trapezoidal and $\mu = 1/2$ is explicit midpoint.

- A one-parameter $s=3$ stage family with order $p=3$ (Slide Set 30 of FCM 1 Fall 20 on slide 19)

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} - \frac{1}{4\mu} & \frac{1}{4\mu} & 0 \\ \hline & \frac{1}{4} & \frac{3}{4} - \mu & \mu \end{array}$$

$$\hat{y}_1 = y_{n-1} \rightarrow \hat{f}_1 = f_{n-1}$$

$$\hat{y}_2 = y_{n-1} + h\left(\frac{2}{3}\,\hat{f}_1\right) \rightarrow \hat{f}_2 = f(t_{n-1} + \frac{2}{3}\,h, \hat{y}_2)$$

$$\hat{y}_3 = y_{n-1} + h\left\{\left(\frac{2}{3} - \frac{1}{4\mu}\right)\,\hat{f}_1 + \frac{1}{4\mu}\,\hat{f}_2\right\} \rightarrow \hat{f}_3 = f(t_{n-1} + \frac{2}{3}\,h, \hat{y}_3)$$

$$y_n = y_{n-1} + h\left\{\frac{1}{4}\,\hat{f}_1 + \left(\frac{3}{4} - \mu\right)\hat{f}_2 + \mu\,\hat{f}_3\right\}$$

- Classical Explicit Runge Kutta 4-stage 4th order: (Slide Set 30 of FCM 1 Fall 20 on slide 20)

$$\begin{aligned}
\hat{y}_1 &= y_{n-1}, \quad f_1 = f(t_{n-1}, \hat{y}_1) \\
\hat{y}_2 &= y_{n-1} + \frac{h}{2} f_1, \quad f_2 = f(t_{n-1/2}, \hat{y}_2) \\
\hat{y}_3 &= y_{n-1} + \frac{h}{2} f_2, \quad f_3 = f(t_{n-1/2}, \hat{y}_3) \\
\hat{y}_4 &= y_{n-1} + h f_3, \quad f_4 = f(t_n, \hat{y}_4) \\
y_n &= y_{n-1} + h\left(\frac{1}{6} f_1 + \frac{1}{3} f_2 + \frac{1}{3} f_3 + \frac{1}{6} f_4\right)
\end{aligned}$$

IVP's and Tasks

(1) Consider at least the following initial value problems (Note that the first and third can also be run as systems of ODEs with appropriate modification.)

$$f(t, y) = \lambda y, \quad y(0) = c \rightarrow y(t) = ce^{\lambda t}$$

$$f = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} \omega y_2 \\ -\omega y_1 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \rightarrow \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} \gamma \sin(\omega t) \\ \gamma \cos(\omega t) \end{pmatrix}$$

$$f = \lambda(y - F(t)) + F'(t) \;\; y(0) = y_0$$
$$y(t) = (y_0 - F(0))e^{\lambda t} + F(t)$$

You are encouraged to select other problems for which you have known solution.

(2) You should select various values of λ and ω, initial conditions, final time value and functions for $F(t)$ and then run the methods with various fixed stepsizes.

(3) Integrate the problems with the various choices using different constant stepsizes and examine the behavior in terms of error (since you know the true solution the

global error can be computed) and stability. Compare the methods in terms of accuracy for different stepsizes and observed rate of convergence as $h \to 0$. You should use total number of evaluations of f as the measure of complexity.

(4) Can you detect any evidence in the behavior that reflects the fact that the methods are of different order?

(5) Do you see behavior consistent with regions of absolution stability discussed in the class notes?

(6) For the two families of methods, is there any significant difference in the behavior you observe for different values of the parameter, e.g., between the explicit midpoint and the explicit trapezoidal rule for the two-stage family?

(7) For problems with known solutions and, if you choose examples without closed forms of the solution you may compare the solutions to those given by well-known standard libaries or environments, e.g., MATLAB's ODE library.