Homework 1 Foundations of Computational Math 2 Spring 2021

These problems are related to the class notes that review vector spaces, matrices, norms and inner products in Set 0.

Problem 1.1

This problem considers three basic vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$.

1.1.a. Prove that $\|.\|_1$ is a vector norm.

1.1.b. Prove that $\|.\|_\infty$ is a vector norm.

1.1.c. Consider $\|.\|_2$.

(i) Show that $\|.\|_2$ is definite.

(ii) Show that $\|.\|_2$ is homogeneous.

(iii) Show that for $\|.\|_2$ the triangle inequality follows from the Cauchy inequality $|x^H y| \leq \|x\|_2 \|y\|_2$.

(iv) Assume you have two vectors x and y such that $\|x\|_2 = \|y\|_2 = 1$ and $x^H y = |x^H y|$, prove the Cauchy inequality holds for x and y.

(v) Assume you have two arbitrary vectors $\tilde{x}$ and $\tilde{y}$. Show that there exists x and y that satisfy the conditions of part (iv) and $\tilde{x} = \alpha x$ and $\tilde{y} = \beta y$ where α and β are scalars.

(vi) Show the Cauchy inequality holds for two arbitrary vectors $\tilde{x}$ and $\tilde{y}$.

Problem 1.2

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, i.e.,

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

.

1.2.a. Suppose you are given a routine that returns $F(x)$ given any $x \in \mathbb{R}^n$. How would you use this routine to determine a matrix $A \in \mathbb{R}^{m \times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$?

1.2.b. Show A is unique.

Problem 1.3

Let $y \in \mathbb{R}^m$ and $\|y\|$ be any vector norm defined on $\mathbb{R}^m$. Let $x \in \mathbb{R}^n$ and A be an $m \times n$ matrix with $m > n$.

1.3.a. Show that the function $f(x) = \|Ax\|$ is a vector norm on $\mathbb{R}^n$ if and only if A has full column rank, i.e., $rank(A) = n$.

1.3.b. Suppose we choose $f(x)$ from part (1.3.a) to be $f(x) = \|Ax\|_2$. What condition on A guarantees that $f(x) = \|x\|_2$ for any vector $x \in \mathbb{R}^n$?

Problem 1.4

Theorem 1. *If $\mathcal{V}$ is a real vector space with a norm $\|v\|$ that satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{1}$$

then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

This problem proves this theorem by a series of lemmas. Prove each of the following lemmas and then prove the theorem.

Lemma 2. $\forall x \in \mathcal{V}$

$$f(x,x) = \|x\|^2$$

Lemma 3. $\forall x, \quad y \in \mathcal{V}$ *$f(x,x)$ is definite and $f(x,y) = f(y,x)$, i.e., (f is symmetric)*

Lemma 4. *The following two "cosine laws" hold $\forall x, \quad y \in \mathcal{V}$:*

$$2f(x,y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \tag{2}$$
$$2f(x,y) = -\|x-y\|^2 + \|x\|^2 + \|y\|^2 \tag{3}$$

Lemma 5. $\forall x, \quad y \in \mathcal{V}$:

$$|f(x,y)| \leq \|x\|\|y\| \tag{4}$$
$$f(x,y) = \gamma\|x\|\|y\|, \quad sign(\gamma) = sign(f(x,y)), \quad 0 \leq |\gamma| \leq 1 \tag{5}$$

Lemma 6. $\forall x, \quad y, \quad z \in \mathcal{V}$:

$$f(x+z,y) = f(x,y) + f(z,y)$$

Lemma 7. $\forall x, \quad y \in \mathcal{V}, \quad \alpha \in \mathbb{R}$

$$f(\alpha x, y) = \alpha f(x,y)$$

Problem 1.5

1.5.a. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$.

1.5.b. Suppose $A \in \mathbb{R}^{m\times n}$ with $m > n$ and let $M \in \mathbb{R}^{n\times n}$ be a nonsingular square matrix. Show that $\mathcal{R}(A) = \mathcal{R}(AM)$ where $\mathcal{R}()$ denotes the range of a matrix.

Problem 1.6

Consider the matrix

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}$$

Suppose that $\lambda_{11} \neq 0$, $\lambda_{33} \neq 0$, $\lambda_{44} \neq 0$ but $\lambda_{22} = 0$.

1.6.a. Show that L is singular.

1.6.b. Determine a basis for the nullspace $\mathcal{N}(L)$.

Problem 1.7

Suppose $A \in \mathbb{C}^{m\times n}$ and let the matrix B be **any submatrix of** A. Show that $\|B\|_p \leq \|A\|_p$.

Problem 1.8

Suppose that $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ and let $E = uv^T$.

1.8.a. Show that $\|E\|_F = \|E\|_2 = \|u\|_2\|v\|_2$.

1.8.b. Show that $\|E\|_\infty = \|u\|_\infty\|v\|_1$.

Problem 1.9

Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two subspaces of $\mathbb{R}^n$.

1.9.a. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Show that x_1 and x_2 are linearly independent.

1.9.b. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Also, suppose that $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $x_3 \neq 0$, i.e., the intersection is not empty. Show that x_1, x_2 and x_3 are linearly independent.

Problem 1.10

Suppose $A \in \mathbb{C}^{m\times n}$. Consider the matrix norm $\|A\|$ induced by the two vector 1-norms $\|x\|_1$ and $\|y\|_1$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ respectively,

$$\|A\| = \max_{\|x\|_1=1} \|Ax\|_1.$$

Is this induced norm the same as the matrix 1-norm defined by

$$\|A\|_1 = \max_{1\leq i\leq n} \|Ae_i\|_1?$$

If so prove it. If not give counterexample to disprove it.

Problem 1.11

Consider the definition of the matrix norm $\|A\| = \max_{i,j} |\alpha_{i,j}|$ where $e_i^T Ae_j = \alpha_{i,j}$.

1.11.a. Show that this defines a matrix norm.

1.11.b. Show that the matrix norm is not consistent.