Homework 3 Foundations of Computational Math 2 Spring 2021

These study questions relate to the material optimization-based iterative methods for linear systems defined by symmetric positive definite matrices.

Problem 3.1

Let x and y be two vectors in $\mathbb{R}^n$.

3.1.a. Show that given x and y the value of $\|x - \alpha y\|_2$ is minimized when

$$\alpha_{min} = \frac{x^T y}{y^T y}$$

3.1.b. Show that $x = y\alpha_{min} + z$ where $y^T z = 0$, i.e., x is easily written as the sum of two orthogonal vectors with specifed minimization properties.

Problem 3.2

Recall that an elementary reflector has the form $Q = I + \alpha z z^T \in \mathbb{R}^{n\times n}$ with $\|z\|_2 \neq 0$.

3.2.a. Show that Q is orthogonal if and only if

$$\alpha = \frac{-2}{z^T z} \text{ or } \alpha = 0$$

3.2.b. Given $v \in \mathbb{R}^n$, let $\gamma = \pm\|v\|$ and $z = v + \gamma e_1$. Assuming that $z \neq v$ show that

$$\frac{z^T z}{z^T v} = 2$$

3.2.c. Using the definitions and results above show that $Qv = -\gamma e_1$

Problem 3.3

Let $x \in \mathbb{R}^n$ be a known vector with components $\xi_i = e_i^T x,\ 1 \leq i \leq n$ and consider the computation of

$$\nu = \xi_1 - \|x\|_2$$

where $\|x\|_2^2 = \sum_{i=1}^n \xi_i^2$. (Recall this is a key computation in the production of a Householder reflector in least squares problems.) When $\xi_1 > 0$ and $\xi_1 \approx \|x\|_2$ the cancellation in the subtraction may result in a significant loss of accuracy.

Find an alternate expression for ν that does not suffer from cancellation when $\xi_1 > 0$ and $\xi_1 \approx \|x\|_2$. (**Hint:** Consider a difference of squares.)

Problem 3.4

Consider a Householder reflector, H, in $\mathbb{R}^2$. Show that

$$H = \begin{pmatrix} -\cos(\phi) & -\sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

where ϕ is some angle.

Problem 3.5

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, $C \in \mathbb{R}^{n\times n}$ be a symmetric nonsingular matrix, and $b \in \mathbb{R}^n$ be a vector. The matrix $M = C^2$ is therefore symmetric positive definite. Also, let $\tilde{A} = C^{-1}AC^{-1}$ and $\tilde{b} = C^{-1}b$.

The preconditioned Steepest Descent algorithm to solve $Ax = b$ is:

> A, M are symmetric positive definite
> x_0 arbitrary; $r_0 = b - Ax_0$; solve $Mz_0 = r_0$
>
> do $k = 0, 1, \ldots$ until convergence
>
> $w_k = Az_k$
> $\alpha_k = \frac{z_k^T r_k}{z_k^T w_k}$
> $x_{k+1} \leftarrow x_k + z_k\alpha_k$
> $r_{k+1} \leftarrow r_k - w_k\alpha_k$
> solve $Mz_{k+1} = r_{k+1}$
>
> end

The Steepest Descent algorithm to solve $\tilde{A}\tilde{x} = \tilde{b}$ is:

> $\tilde{A}$ is symmetric positive definite
> $\tilde{x}_0$ arbitrary; $\tilde{r}_0 = \tilde{b} - \tilde{A}\tilde{x}_0$; $\tilde{v}_0 = \tilde{A}\tilde{r}_0$
>
> do $k = 0, 1, \ldots$ until convergence
>
> $\tilde{\alpha}_k = \frac{\tilde{r}_k^T \tilde{r}_k}{\tilde{r}_k^T \tilde{v}_k}$
> $\tilde{x}_{k+1} \leftarrow \tilde{x}_k + \tilde{r}_k\tilde{\alpha}_k$
> $\tilde{r}_{k+1} \leftarrow \tilde{r}_k - \tilde{v}_k\tilde{\alpha}_k$
> $\tilde{v}_{k+1} \leftarrow \tilde{A}\tilde{r}_{k+1}$
>
> end

Show that given the appropriate consistency between initial guesses the preconditioned steepest descent recurrences to solve $Ax = b$ can be derived from the steepest descent recurrences to solve $\tilde{A}\tilde{x} = \tilde{b}$.

Problem 3.6

Consider solving a linear system $Ax = b$ where A is symmetric positive definite using steepest descent.

3.6.a

Suppose you use steepest descent without preconditioning. Show that the residuals, r_k and r_{k+1} are orthogonal for all k.

3.6.b

Suppose you use steepest descent with preconditioning. Are the residuals, r_k and r_{k+1} orthogonal for all k? If not is there any vector from step k that is guaranteed to be orthogonal to r_{k+1}?

Problem 3.7

Suppose $A \in \mathbb{R}^{n\times n}$ is a **symmetric positive semidefinite** matrix and $f(x) = 0.5x^T Ax - x^T b$ with $b \in \mathbb{R}^n$ and $b \in \mathcal{R}(A)$. Show that Steepest Descent will converge to an unconstrained minimizer of $f(x)$ for any x_0 such that $Ax_0 \neq 0$.

Hint: Find a smaller, symmetric positive definite linear system and use the fact that steepest descent converges on a symmetric positive definite system.

Problem 3.8

Let $A = Q\Lambda Q^T$ be a symmetric positive definite matrix where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are positive and also are the eigenvalues of A. Define

$$\tilde{x} = Q^T x \quad \text{and} \quad \tilde{b} = Q^T b$$
$$Ax = b \quad \text{and} \quad \Lambda\tilde{x} = \tilde{b}$$

Given x_0 and $\tilde{x}_0$, define the sequence x_k as the sequence of vectors produced by applying CG to solve $Ax = b$ and the sequence $\tilde{x}_k$ as the sequence of vectors produced by applying CG to solve $\Lambda\tilde{x} = \tilde{b}$.

Let $e_k = x_k - x$ and $\tilde{e}_k = \tilde{x}_k - \tilde{x}$. Show that if $\tilde{x}_0 = Q^T x_0$ then

$$\|e_k\|_2 = \|\tilde{e}_k\|_2, \quad k > 0$$