Homework 5 Foundations of Computational Math 2 Spring 2021

These study questions relate to the material on solving scalar and systems of nonlinear equations.

Problem 5.1

Let $\phi(x) : [a,b] \to [a,b]$ be a continuous function. Show that if $\phi(x)$ is a contraction mapping on $[a,b]$ then the sequence $\{x^{(k)}\}$ defined by $x^{(k+1)} = \phi(x^{(k)})$ is a Cauchy sequence.

Problem 5.2

Let $f(x) = x^3 - 3x + 1$. This polynomial has three distinct roots.

(5.2.a) Consider using the iteration function

$$\phi_1(x) = \frac{1}{3}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_1(x)$ and how would you choose $x^{(0)}$ for each computable root?

(5.2.b) Consider using the iteration function

$$\phi_2(x) = \frac{3}{2}x - \frac{1}{6}(x^3 + 1)$$

Which, if any, of the three roots can you compute with $\phi_2(x)$ and how would you choose $x^{(0)}$ for each computable root?

(5.2.c) For each of the roots you identified as computable using either $\phi_1(x)$ or $\phi_2(x)$, apply the iteration to find the values of the roots. (You need not turn in any code, but using a simple program to do this is recommended.)

Problem 5.3

Textbook, p. 283, Problem 2

Problem 5.4

Textbook, p. 283, Problem 6

Problem 5.5

Textbook, p. 284, Problem 8

Problem 5.6

5.6.a

Consider the polynomial

$$p(x) = x^3 - x - 5$$

The following three iterations can be derived from $p(x)$.

$$\phi_1(x) = x^3 - 5$$
$$\phi_2(x) = \sqrt[3]{x+5}$$
$$\phi_3(x) = 5/(x^2 - 1)$$

The polynomial $p(x)$ has a root $r > 1.5$. Determine which of the $\phi_i(x)$ produce an iteration that converges to r.

5.6.b

Consider the function

$$f(x) = xe^{-x} - 0.06064$$

5.6.a. Write the update formula for Netwon's method to find the root of $f(x)$.

5.6.b. $f(x)$ has a root of $\alpha = 0.06469263\ldots$. If you run Newton's method with $x_0 = 0$ convergence occurs very quickly, e.g., in double precision $|f(x_4)| \approx 10^{-10}$. However, if x_0 is large and negative or if x_0 is near 1 convergence is much slower until there is a very rapid improvement in accuracy in the last one or two steps. For example, if $x_0 = 0.98$ then

$$|f(x_{47})| \approx 0.6 \times 10^{-2}$$
$$|f(x_{49})| \approx 1.5 \times 10^{-5}$$
$$|f(x_{50})| \approx 2.8 \times 10^{-10}$$

Explain this behavior. You might find it useful to plot $f(x)$ and run a few examples of Newton's method.

Problem 5.7

Consider the fixed point iteration by the function

$$\phi(x) = x - \frac{(x^2 - 3)}{(x^2 + 2x - 3)}$$

The value $\alpha = \sqrt{3}$ is a fixed point for this iteration. Provide justification to all of your answers for the following:

(5.7.a) Show that there exists a nontrivial interval $\alpha - \delta < x < \alpha + \delta$ with $\delta > 0$ such that the iteration defined by $\phi(x)$ converges to α for any x_0 in the interval.

(5.7.b) Is the order of convergence on this interval linear ($p = 1$) or higher ($p \geq 2$)?

(5.7.c) Show that for $x > \alpha$ we have

$$\alpha < \phi(x) < x$$

and use this to explain the convergence of the iteration when $x_0 > \alpha$.

(5.7.d) Plot or otherwise enumerate values of the curves $y = \phi(x)$ and $y = x$ on the interval $0 < x < \beta$ for $\beta > \alpha$. Use the information to examine the behavior of the iteration for $0 < x_0 < \alpha$. For what subinterval, if any, do you expect convergence to α?

You need not turn in the plot or enumeration. Simply state the important characteristics that support your conclusion. The plot can also assist in supporting your answers for the earlier parts of the question.

Problem 5.8

Textbook, page 330, Problem 6

Problem 5.9

5.9.a

Consider the system of equations in $\mathbb{R}^2$

$$\xi^2 + \eta^2 = 4$$
$$e^{\xi} + \eta = 1$$

Show that the system has two solutions in $\mathbb{R}^2$, one with $\xi > 0$ and $\eta < 0$ and one with $\xi < 0$ and $\eta > 0$.

5.9.b

Consider the following function and iteration:

$$G_1(x) = \begin{pmatrix} \gamma_1(\xi,\eta) \\ \gamma_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} \ln(1-\eta) \\ -\sqrt{4-\xi^2} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k,\eta_k) \\ \gamma_2(\xi_k,\eta_k) \end{pmatrix} = \begin{pmatrix} \ln(1-\eta_k) \\ -\sqrt{4-\xi_k^2} \end{pmatrix}$$

(i) Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.

(ii) Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.

(iii) Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.

(iv) Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.

5.9.c

Consider the following function and iteration:

$$G_2(x) = \begin{pmatrix} \gamma_1(\xi,\eta) \\ \gamma_2(\xi,\eta) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta^2} \\ 1-e^{\xi} \end{pmatrix}$$

$$\begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} = \begin{pmatrix} \gamma_1(\xi_k,\eta_k) \\ \gamma_2(\xi_k,\eta_k) \end{pmatrix} = \begin{pmatrix} -\sqrt{4-\eta_k^2} \\ 1-e^{\xi_k} \end{pmatrix}$$

(i) Note that all iterates must remain real. Consider where in the $\mathbb{R}^2$ a value of ξ_k or η_k would cause complex values to appear on the next iteration.

(ii) Find a domain for the initial condition $(\xi_0,\ \eta_0)$ such that the iteration converges to one of the roots. Prove your assertion.

(iii) Can this iteration converge to either root by appropriate choice of $(\xi_0,\ \eta_0)$? Prove your assertion.

(iv) Implement the iteration and verify with several initial conditions in the domain that iteration converges to the predicted root.