Stationary Iterative Methods Study Questions Foundations of Computational Mathematics 2 Fall 2020

Problem 0.1

0.1.a. Textbook page 241, Problem 2

0.1.b. Textbook page 241, Problem 4

0.1.c. Textbook page 241, Problem 5

Material in textbook Sections 1.7 and 5.1 is useful for these problems.

Problem 0.2

The Gershgorin theorems in Section 5.1 of the text and the idea of irreducibility in Section 5.1 are often valuable in analyzing the convergence of iterative methods. Familiarize yourself with both in order to answer this question.

0.2.a. Use the appropriate Gershgorin-related facts to show that if A is symmetric and strictly diagonally dominant then Jacobi converges for all x_0.

0.2.b. Use the appropriate Gershgorin-related facts to show that Jacobi converges for all x_0 for the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Problem 0.3

0.3.a

Consider the two matrices:

$$A_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad and \quad A_2 = \begin{pmatrix} 1 & -\frac{1}{12} \\ -\frac{3}{4} & 1 \end{pmatrix}$$

Suppose you solve systems of linear equations involving A_1 and A_2 using Jacobi's method. For which matrix would you expect faster convergence?

0.3.b

Consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}$$

(i) Will Jacobi's method converge when solving $Ax = b$?

(ii) Will Gauss-Seidel converge when solving $Ax = b$?

Problem 0.4

Consider simple accelerated Richardson's method to solve $Ax = b$

$$\begin{gathered} \text{Given,} \quad x_0 \\ x_{k+1} = x_k + \alpha r_k \\ r_k = b - Ax_k \\ \alpha > 0 \end{gathered}$$

The nonsymmetric matrix

$$A = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

has real positive eigenvalues. Computing the eigenvalues A or the matrix that defines the iteration is too costly in practice. What value would choose for $\alpha > 0$ so that simple accelerated Richardson's method will converge for any x_0? Make sure your method for setting α is practical.

Problem 0.5

0.5.a

Consider solving $Ax = b$ where the matrix A is nonsingular by a linear stationary iterative method

$$x_{k+1} = Gx_k + f, \quad G = (I - P^{-1}A), \quad f = P^{-1}b$$

(i) Give an example of an iterative method for which G is singular. Justify your answer.

(ii) Does G being singular affect the asymptotic rate of the iteration compared to another iteration defined by $\tilde{G}$, that differs only in that $\tilde{G}$ has a nonzero eigenvalue when G has a zero eigenvalue with the rest of the eigenvalues the same for both matrices?

0.5.b

When attempting to solve $Ax = b$ where A is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on A to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix A. Let P and Q be two permutation matrices. Rather than solving $Ax = b$ we could solve $(PAQ)(Q^T x) = Pb$ using an iterative method. Sometimes it is possible to examine A and choose P and/or Q so that it is easy to apply one of our sufficient condition theorems.

(i) Can you choose P and/or Q so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & -2 & 7 \\ 1 & 6 & -1 \\ 10 & -2 & 7 \end{pmatrix}?$$

(ii) Can you choose P and/or Q so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}?$$

Problem 0.6

Determine the necessary and sufficient conditions for $x = A^{-1}b$ to be a fixed point of

$$x_{i+1} = Gx_i + f$$

Problem 0.7

Prove the following:

Lemma 0.7.1. *If A can be written $A = I - P$ where $P \geq 0$ and $\rho(P) < 1$ then A is an M-matrix.*

Problem 0.8

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider solving the linear system $Ax = b$ using a linear stationary method with preconditioner P. Also assume that P is symmetric positive definite with a Cholesky factorization $P = LL^T$.

Show that Richardson's stationary method of the form

$$x_{k+1} = x_k + P^{-1} r_k$$

converges for all x_0 if the symmetric matrix $2P - A$ is positive definite.

Problem 0.9

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider solving the linear system $Ax = b$.

Recall that for Symmetric Gauss-Seidel we have

$$P_{sgs} = (D - L) D^{-1} (D - U)$$

$$x_{k+1} = x_k + P_{sgs}^{-1} r_k = x_k + (D - U)^{-1} D (D - L)^{-1} r_k.$$

Show that the Symmetric Gauss-Seidel iteration converges for any x_0.

Problem 0.10

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider solving the linear system $Ax = b$. Show that the SSOR iteration converges for any x_0 when $0 < \omega < 2$.

Problem 0.11

All parts of this problem are related to the linear system

$$Ax = b$$

where $A \in \mathbb{R}^{n \times n}$ is the matrix

$$A = \begin{pmatrix} 2 & \alpha & & & & & & \\ \alpha & 3 & \alpha & & & & & \\ & \alpha & 4 & \alpha & & & & \\ & & \alpha & 5 & \alpha & & & \\ & & & \alpha & 6 & \alpha & & \\ & & & & \alpha & 7 & \alpha & \\ & & & & & \alpha & 8 & \alpha \\ & & & & & & \alpha & 9 \end{pmatrix}$$

$-1 < \alpha < 0$. and $b \in \mathbb{R}^n$.

0.11.a

State several properties of the matrix A that are of interest when solving $Ax = b$ using the iterative methods we have discussed. Justify your claims.

0.11.b

Which of the following methods converge for any x_0 when applied to $Ax = b$?

- Jacobi
- Gauss-Seidel
- Symmetric Gauss-Seidel

Justify your answers.

0.11.c

Determine the number of operations required for one step of Jacobi and one step of Gauss-Seidel for $Ax = b$. Express your answers as

$$Cn^d + O(n^{d-1})$$

by giving C and d for both methods.

0.11.d

Suppose Jacobi and Gauss-Seidel are started with the same initial condition x_0 and both are run until the initial residual is reduced to some specified level. Do you expect either method to require significantly fewer total computations than the other method? Justify your answer.

Problem 0.12

Suppose the system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ is to be solved using the (Forward) Gauss-Seidel method.

Show that if A is strictly diagonally dominant by rows then the method will converge.