Homework 5 Foundations of Computational Mathematics 2 Fall 2019

Problem 5.1

5.1.a. Textbook page 241, Problem 2

5.1.b. Textbook page 241, Problem 4

5.1.c. Textbook page 241, Problem 5

Material in textbook Sections 1.7 and 5.1 is useful for these problems.

Problem 5.2

The Gershgorin theorems in Section 5.1 of the text and the idea of irreducibility in Section 5.1 are often valuable in analyzing the convergence of iterative methods. Familiarize yourself with both in order to answer this question.

5.2.a. Use the appropriate Gershgorin-related facts to show that if A is symmetric and strictly diagonally dominant then Jacobi converges for all x_0.

5.2.b. Use the appropriate Gershgorin-related facts to show that Jacobi converges for all x_0 for the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Problem 5.3

5.3.a

Consider the two matrices:

$$A_1 = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \quad \textit{and} \quad A_2 = \begin{pmatrix} 1 & -\frac{1}{12} \\ -\frac{3}{4} & 1 \end{pmatrix}$$

Suppose you solve systems of linear equations involving A_1 and A_2 using Jacobi's method. For which matrix would you expect faster convergence?

5.3.b

Consider the matrix

$$A = \begin{pmatrix} 4 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 4 \end{pmatrix}$$

(i) Will Jacobi's method converge when solving $Ax = b$?

(ii) Will Gauss-Seidel converge when solving $Ax = b$?

Problem 5.4

Consider simple accelerated Richardson's method to solve $Ax = b$

$$\begin{aligned} &\text{Given,} \quad x_0 \\ &x_{k+1} = x_k + \alpha r_k \\ &r_k = b - Ax_k \\ &\alpha > 0 \end{aligned}$$

The nonsymmetric matrix

$$A = \begin{pmatrix} 10 & 2 & 3 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

has real positive eigenvalues. What value would choose for $\alpha > 0$ so that simple accelerated Richardson's method will converge for any x_0? Justify your answer.

Problem 5.5

5.5.a

Consider solving $Ax = b$ where the matrix A is nonsingular by a linear stationary iterative method

$$x_{k+1} = Gx_k + f, \quad G = (I - P^{-1}A), \quad f = P^{-1}b$$

(i) Give an example of an iterative method for which G is singular. Justify your answer.

(ii) Does G being singular affect the asymptotic rate of the iteration compared to another iteration defined by $\tilde{G}$, that differs only in that $\tilde{G}$ has a nonzero eigenvalue when G has a zero eigenvalue with the rest of the eigenvalues the same for both matrices?

5.5.b

When attempting to solve $Ax = b$ where A is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on A to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix A. Let P and Q be two permutation matrices. Rather than solving $Ax = b$ we could solve $(PAQ)(Q^T x) = Pb$ using an iterative method. Sometimes it is possible to examine A and choose P and/or Q so that it is easy to apply one of our sufficient condition theorems.

(i) Can you choose P and/or Q so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & -2 & 7 \\ 1 & 6 & -1 \\ 10 & -2 & 7 \end{pmatrix}?$$

(ii) Can you choose P and/or Q so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}?$$

Problem 5.6

Determine the necessary and sufficient conditions for $x = A^{-1}b$ to be a fixed point of

$$x_{i+1} = Gx_i + f$$

Problem 5.7

Suppose you are attempting to solve $Ax = b$ using a linear stationary iterative method defined by

$$x_k = Gx_{k-1} + f$$

that is consistent with $Ax = b$, i.e., for which $f = (I - G)A^{-1}b$.

Suppose the eigenvalues of G are real and such that $|\lambda_1| > 1$ and $|\lambda_i| < 1$ for $2 \leq i \leq n$. Also suppose that G has n linearly independent eigenvectors, z_i, $1 \leq i \leq n$.

5.7.a. Show that there exists an initial condition x_0 such that x_k converges to $x = A^{-1}b$.

5.7.b. Does your answer give a characterization of selecting x_0 that could be used in practice to create an algorithm that would ensure convergence?

Problem 5.8

Prove the following:

Lemma 5.8.1. *If A can be written $A = I - P$ where $P \geq 0$ and $\rho(P) < 1$ then A is an M-matrix.*

Problem 5.9

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider solving the linear system $Ax = b$ using a linear stationary method with preconditioner P. Assume that P is also symmetric positive definite with a Cholesky factorization $P = LL^T$.

This question shows that Richardson's stationary method of the form

$$x_{k+1} = x_k + P^{-1}r_k$$

converges for all x_0 if and only if the symmetric matrix $2P - A$ is positive definite.

5.9.a

The iteration matrix is $G = I - P^{-1}A$. Show that G is similar to the symmetric matrix $I - \tilde{A}$ where $\tilde{A}$ is symmetric positive definite and is defined by using A and matrices related to L.

5.9.b

Show that that if $\rho(I - \tilde{A}) < 1$ then the eigenvalues of the symmetric positive definite matrix $\tilde{A}$ must satisfy

$$0 < \tilde{\lambda} < 2.$$

5.9.c

For any symmetric positive definite matrix, M, we have that

$$\forall v \neq 0, \quad 0 < \frac{v^T M v}{v^T v} \leq \lambda_{max}$$

with equality when v is an eigenvector of λ_{max}, i.e.,

$$\lambda_{max} = \max_{v \neq 0 \in \mathbb{R}^n} \frac{v^T M v}{v^T v}.$$

Show that that if $\rho(I - \tilde{A}) < 1$ then the symmetric matrix $2P - A$ is positive definite.

5.9.d

Show the reverse direction, i.e., show that if the symmetric matrix $2P - A$ is positive definite then $\rho(I - \tilde{A}) < 1$.

Hint: this simply uses all that has been deduced above.

Problem 5.10

5.10.a

Let $A = D - L - U \in \mathbb{R}^{n \times n}$ be a nonsingular matrix, where $-L$ is the matrix of strictly lower triangular elements and $-U$ is the matrix of strictly upper triangular elements. Recall that the three methods

- Gauss-Seidel (forward): $P_{gs} = D - L$

$$x_{k+1} = x_k + P_{gs}^{-1} r_k = x_k + (D - L)^{-1} r_k$$

- Gauss-Seidel (backward): $P_{bgs} = D - U$

$$x_{k+1} = x_k + P_{bgs}^{-1} r_k = x_k + (D - U)^{-1} r_k$$

- Symmetric Gauss-Seidel: $P_{sgs} = (D - L) D^{-1} (D - U)$

$$x_{k+1} = x_k + P_{sgs}^{-1} r_k = x_k + (D - U)^{-1} D (D - L)^{-1} r_k$$

Show that one iteration of Symmetric Gauss-Seidel is equivalent to one iteration of forward Gauss-Seidel followed by one iteration of backward Gauss-Seidel.

5.10.b

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider solving the linear system $Ax = b$. Show that the Symmetric Gauss-Seidel iteration converges for any x_0.

Problem 5.11

Suppose the system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ is to be solved using the (Forward) Gauss-Seidel method.

Show that if A is strictly diagonally dominant by rows then the method will converge.