Study Questions Homework 2 Foundations of Computational Math 2 Spring 2022

Problem 2.1

Assuming that the nodes are uniformly spaced, we have derived the form of the cubic B-spline $B_{3,i}(t)$ and determined its values and the values of $B'_{3,i}(t)$ and $B''_{3,i}(t)$ at the nodes t_{i-2}, t_{i-1}, t_{i-1}, t_i, t_{i+1}, and t_{i+2}. We also derived $B_{1,i}(t)$ and saw that it was the familiar hat function.

2.1.a. Derive the formula of the quadratic B-spline $B_{2,i}(t)$ and determine its values and the values of $B'_{2,i}(t)$ and $B''_{2,i}(t)$ at the appropriate nodes.

2.1.b. Derive the formula of the quintic B-spline $B_{5,i}(t)$ and determine its values and the values of $B'_{5,i}(t)$ and $B''_{5,i}(t)$ at the appropriate nodes.

Problem 2.2

Consider a set of equidistant mesh points, $x_k = x_0 + kh$, $0 \leq k \leq m$ and the following interpolating constraints to define a cubic spline based at the point x_i.
Basic Interpolation Conditions:

$$\begin{aligned} b_i(x_i) &= 1 \\ b_i(x_j) &= 0, \quad 0 \leq j \leq i-2 \\ b_i(x_j) &= 0, \quad i+2 \leq j \leq n \end{aligned}$$

Note **we do not require** $b_i(x_j) = 0$ for $j = i-1$ and $j = i+1$. Therefore, we need four boundary conditions to complete the definition:

$$\begin{aligned} b'_i(x_{i-2}) &= 0 \\ b''_i(x_{i-2}) &= 0 \\ b'_i(x_{i+2}) &= 0 \\ b''_i(x_{i+2}) &= 0 \end{aligned}$$

Derive a system of equations that shows that the cubic spline $b_i(x)$ is a scaled version of the cubic B-spline $B_i(x)$ defined in the notes and textbook.

Problem 2.3

Consider an interpolatory quadratic spline, $s(x)$, that satisfies the following interpolation conditions and single boundary condition:

$$s(x_i) = f(x_i) = f_i, \quad 0 \le i \le n$$

$$s'(x_0) = f'(x_0) = f'_0$$

where the x_i are distinct.

2.3.a. Derive a linear system of equations that yields the values

$$s'(x_i) = s'_i \quad 0 \le i \le n$$

that are used as parameters to define the quadratic spline $s(x)$.

2.3.b. Identify important structure in the linear system and show that it defines a unique quadratic spline.

2.3.c. Use the structure of the system to show that if $f(x)$ is a quadratic polynomial then $s(x) = f(x)$.

Problem 2.4

Recall when defining an interpolatory cubic spline $s(t)$ in terms of the parameters s''_i for $0 \le i \le n$, we have the $(n-1) \times (n-1)$ linear system

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \dots & 0 \\ \mu_2 & 2 & \lambda_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \mu_{n-2} & 2 & \lambda_{n-2} \\ 0 & \dots & 0 & \mu_{n-1} & 2 \end{pmatrix} \begin{pmatrix} s''_1 \\ \vdots \\ s''_{n-1} \end{pmatrix} = \begin{pmatrix} d_1 - \mu_1 s''_0 \\ \vdots \\ d_{n-1} - \lambda_{n-1} s''_n \end{pmatrix}$$

when the boundary conditions $s''_0 = c_0$ and $s''_n = c_n$ are specified, where c_0 and c_n are given constants and

$$h_i = x_i - x_{i-1}, \quad \mu_i = \frac{h_i}{h_i + h_{i+1}}$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad d_i = \frac{6}{h_i + h_{i+1}} \left(f[i, i+1] - f[i-1, i] \right).$$

Derive the additional equations, in terms of the s''_i parameterization, defining a unique cubic spline when the boundary conditions

$$s'_0 = c_0 \quad \text{and} \quad s'_n = c_n$$

are specified.

Problem 2.5

2.5.a. Suppose you are given an arbitrary polynomial of degree 3 or less with the form

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3.$$

Show that there are unique coefficients, γ_i, $0 \le i \le 3$, for $p(x)$ in the representation of the form

$$p(x) = \gamma_0 T_0(x) + \gamma_1 T_1(x) + \gamma_2 T_2(x) + \gamma_3 T_3(x)$$

where $T_i(x)$, $0 \le i \le 3$, are the Chebyshev polynomials.

2.5.b. Is this true for any degree n? Justify your answer.

2.5.c. Consider $T_{32}(x)$, the Chebyshev polynomial of degree 32 and $T_{51}(x)$, the Chebyshev polynomial of degree 51. What is the coefficient of x^{13} in $T_{32}(x)$? What is the coefficient of x^{20} in $T_{51}(x)$?

Problem 2.6

For this problem, consider the space $\mathcal{L}^2[-1,1]$ with inner product and norm

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \text{ and } \|f\|^2 = (f,f)$$

Let $P_i(x)$, for $i = 0, 1, \ldots$ be the Legendre polynomials of degree i and let $n+1-$st have the form

$$P_{n+1}(x) = \rho_n (x - x_0)(x - x_1)\cdots(x - x_n)$$

i.e., x_i for $0 \le i \le n$ are the roots of $P_{n+1}(x)$.

Let the Lagrange interpolation functions that use the x_i be $\ell_i(x)$ for $0 \le i \le n$. So, for example,

$$L_n(x) = \ell_0(x)f(x_0) + \cdots + \ell_n(x)f(x_n)$$

is the Lagrange form of the interpolation polynomial of $f(x)$ defined by the roots.

Let $\mathbb{P}_n$ be the space of polynomials of degree less than or equal to n. We can write the least squares approximation of $f(x)$ in terms of the $P_i(x)$ using the generalized Fourier series as

$$f_n(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_n P_n(x) \text{ where } \alpha_i = \frac{(f, P_i)}{(P_i, P_i)}$$

2.6.a

Clearly, $(\ell_i, \ell_i) \neq 0$. Show that $(\ell_i, \ell_j) = 0$ when $i \neq j$. Therefore, the functions $\ell_0(x), \ldots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$.

2.6.b

Suppose we evaluate $f_n(x)$ at the x_i to obtain the data $f_n(x_0), \ldots, f_n(x_n)$. We can then write $f_n(x)$ in its Lagrange form,

$$f_n(x) = L_n(x) = f_n(x_0)\ell_0(x) + \ldots + f_n(x_n)\ell_n(x)$$

Since the $\ell_0(x), \ldots, \ell_n(x)$ are an orthogonal basis for $\mathbb{P}_n$, they also can be used to compute, $f_n(x)$, the unique least squares approximation to $f(x)$. As with the Legendre polynomials, using the generalized Fourier series, yields

$$f_n(x) = \sigma_0\ell_0(x) + \sigma_1\ell_1(x) + \cdots + \sigma_n\ell_n(x) \text{ where } \sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)}$$

Show that these two forms of $f_n(x)$ give the same polynomial by showing that

$$\sigma_i = \frac{(f, \ell_i)}{(\ell_i, \ell_i)} = f_n(x_i)$$

Hint: Consider the relationship between $f(x)$ and $f_n(x)$.

Problem 2.7

Consider $f(x) = e^x$ on the interval $-1 \leq x \leq 1$. Suppose we want to approximate $f(x)$ with a polynomial. Generate the following polynomials:

(a) $F_1(x)$ and $F_3(x)$:the first and third order Taylor series approximations of $f(x)$ expanded about $x = 0$.

(b) $N_1(x)$: the linear near-minimax approximation to $f(x)$ on the interval.

(c) $C_1(x)$ and $C_2(x)$ – the linear and quadratic polynomials that result from Chebyshev economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(d) $p_1(x)$ and $p_2(x)$ – the linear and quadratic polynomials that result from Legendre economization applied to $F_3(x)$, the third order Taylor series approximation of $f(x)$ expanded about $x = 0$.

(2.7.a) Derive bounds on the ∞ norm of the error where possible.

(2.7.b) Evaluate the error for each polynomial approximation on a very fine grid on the interval $-1 \leq x \leq 1$ and compare to the bounds.

Problem 2.8

This is a study question not a programming assignment and you need not turn in any code. This problem considers the use of discrete least squares for approximation by a polynomial. Recall, the distinct points $x_0 < x_1 < \cdots < x_m$ are given and the **discrete** metric

$$c(p_n) = \sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2$$

with $\omega_i > 0$ is used to determine the polynomial, $p_n^{ls}(x)$, of degree n that achieves the minimal value.

Assume that $\omega_i = 1$ for this exercise.

This means that

$$c(p_n) = \sum_{i=0}^{m} \omega_i (f(x_i) - p_n(x_i))^2 = \|F - P_n\|_2^2,$$

where, $p_n(x) \in \mathbb{P}_n$,

$$F = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} \in \mathbb{R}^{m+1}, \quad \text{and} \quad P_n = \begin{pmatrix} p_n(x_0) \\ p_n(x_1) \\ \vdots \\ p_n(x_m) \end{pmatrix} \in \mathbb{R}^{m+1}$$

and the norm is the standard Euclidean norm, i.e., the 2-norm, $\forall v \in \mathbb{R}^{m+1}, \ \|v\|_2^2 = v^T v$.

Typically, $m \gg n$. If $m = n$ then the unique interpolating polynomial is the solution.

Since the optimization problem is over all $p_n \in \mathbb{P}_n$, if we parameterize $\mathbb{P}_n$ as

$$p_n(x) = \sum_{j=0}^{n} \phi_j(x)\gamma_j$$

then the conditions are

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_m \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{pmatrix} - \begin{pmatrix} \phi_0(x_0) & \dots & \phi_n(x_0) \\ \phi_0(x_1) & \dots & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_0(x_m) & \dots & \phi_n(x_m) \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$$r = b - Ag$$

and the optimization problem becomes

$$\min_{g \in \mathbb{R}^{n+1}} \|b - Ag\|_2^2,$$

i.e., minimize the residual r as a function of $g \in \mathbb{R}^{n+1}$.

Use the Chebyshev polynomials to form an orthonormal basis, i.e.,

$$\phi_i(x) = \alpha_i T_i(x)$$

and the roots of $T_{m+1}(x)$ as the x_i.

1. Identify the important property that the matrix possesses that allows the system to be be solved in $O(n)$ comptuations.

2. Verify empirically that the matrix satisfies the property above to numerical precision.

3. Use your solution to implement a code that assembles the least squares problem and solves it to find the optimal solution $g_* \in \mathbb{R}^{n+1}$ Make sure to exploit the algebraic properties of the matrix A to have an efficient solution.

4. Apply your code to several $f(x)$ choices and use multiple n and m values to explore the accuracy of the approximation. Approximate $\|f - p_n^{ls}\|_\infty$ by sampling the difference between f and the polynomial at a large number of points in the interval and taking the maximum magnitude.

5. For each, problem you solve check the residual of the overdetermined system $r = (b - Ag_*$ where g_* is the optimal set of coefficients. Empirically evaluate how it relates to the subspace $\mathcal{R}(A)$.

Problem 2.9

For this problem, consider the space $\mathcal{L}^2[-1,1]$ with inner product and norm

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \text{ and } \|f\|^2 = (f,f)$$

Let $f(x) = x^3 + x^2$. Determine, $p_1(x)$, the best linear least squares fit to $f(x)$ on $\mathcal{L}^2[-1,1]$ with the inner product (f,g), i.e., the linear polynomial that solves

$$\min_{p \in \mathbb{P}_1} \|f(x) - p(x)\|^2$$

where the norm is as defined above.

Problem 2.10

Consider a minimax approximation to a function $f(x)$ on $[a,b]$. Assume that $f(x)$ is continuous with continuous first and second order derivatives. Also, assume that $f''(x) < 0$ on for $a \leq x \leq b$, i.e., f is concave on the interval.

2.10.a. Derive the equations you would solve to determine the linear minimax approximation, $p_1(x) = \alpha x + \beta$, to $f(x)$ on $[a, b]$ and describe their use to solve the problem.

2.10.b. Apply your approach to determine $p_1(x) = \alpha x + \beta$ for $f(x) = -x^2$ on $[-1, 1]$.

2.10.c. How does $p_1(x)$ relate to the quadratic monic Chebyshev polynomial $t_2(x)$?

2.10.d. Apply your approach to determine $\tilde{p}_1(x) = \tilde{\alpha} x + \tilde{\beta}$ for $f(x) = -x^2$ on $[0, 1]$.

2.10.e. How could the quadratic monic Chebyshev polynomial $t_2(y)$ on $-1 \leq y \leq 1$ be used to provide and alternative derivation of $\tilde{p}_1(x)$ on $0 \leq x \leq 1$?

2.10.f. Suppose you adapt your approach to derive a constant approximation, $p_0(x)$. What points will you use as the extrema of the error?

Problem 2.11

Suppose you are given the following analytical information about a function $f(x)$ on $[-1, 1]$:

$$\begin{aligned} f(x) &= \frac{1}{x+3} \\ f'(x) &= \frac{-1}{(x+3)^2} \\ f''(x) &= \frac{2}{(x+3)^3} \end{aligned}$$

2.11.a

Find, $p_1(x)$, the linear polynomial that is the near-minimax approximation to $f(x)$ on the interval $[-1, 1]$.

2.11.b

Find, $q_1(x)$, the linear polynomial that is the minimax (best) approximation to $f(x)$ on the interval $[-1, 1]$.

2.11.c

Give a bound for the error $|f(x) - p_1(x)|$ on the interval $[-1, 1]$.

2.11.d

Give a bound for the error $|f(x) - q_1(x)|$ on the interval $[-1, 1]$.