Study Questions Homework 3 Foundations of Computational Math 2 Spring 2022

Problem 3.1

Consider the quadrature formula

$$I_0(f) = (b-a)f(a) \approx \int_a^b f(x)dx = \mathcal{I}(f)$$

- What is the degree of exactness?
- What is the order of infinitesimal?

Problem 3.2

Consider the two quadrature formulas

$$I_2(f) = \frac{2}{3}\left[2f(-1/2) - f(0) + 2f(1/2)\right]$$
$$I_4(f) = \frac{1}{4}\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right]$$

- What is the degree of exactness when $I_2(f)$ is used to approximate $\mathcal{I}(f;-1,1) = \int_{-1}^{1} f(x)dx$?
- What is the degree of exactness when $I_2(f)$ is used to approximate $\mathcal{I}(f; = 0.5, 0.5)) = \int_{-1/2}^{1/2} f(x)dx$?
- What is the degree of exactness when $I_4(f)$ is used to approximate $\mathcal{I}(f;-1,1) = \int_{-1}^{1} f(x)dx$?

Problem 3.3

Consider

$$\int_a^b \omega(x)f(x)dx \approx \alpha f(x_0)$$

where $\omega(x) = \sqrt{x}$ and $0 \leq a < b$.

Determine α and x_0 such that the degree of exactness is maximized.

Problem 3.4

In this problem we consider the numerical approximation of the integral

$$\mathcal{I}(f;-1,1)=\int_{-1}^{1} f(x)dx$$

with $f(x)=e^x$. In particular, we use a priori error estimation to choose a step size h for Newton Cotes or a number of points for a Gaussian integration method.

3.4.a

Consider the use of the composite Trapezoidal rule to approximate the integral $\mathcal{I}(f;-1,1)$.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error using the composite trapezoidal rule and to determine a stepsize h so that the error will be less than or equal to the tolerance 10^{-2}.
- Approximately how many points does your h require?

3.4.b

Consider the use of the Gauss-Legendre method to approximate the integral $\mathcal{I}(f;-1,1)$. Use $n=1$, i.e., two points x_0 and x_1 with weights γ_0 and γ_1.

- Use the fact that we have an analytical form of $f(x)$ to estimate the error that will result from using the two-point Gauss-Legendre method to approximate the integral.
- How does your estimate compare to the tolerance 10^{-2} used in the first part of the question?
- Recall that for $n=1$ we have the Gauss Legendre nodes $x_0 \approx -0.5774$ and $x_1 \approx 0.5774$. Apply the method to approximate I and compare its error to your prediction. The true value is

$$\mathcal{I}(f;-1,1)=\int_{-1}^{1} e^x dx \approx 2.3504$$

Problem 3.5

Let $U(x)$ and $V(x)$ be polynomials of degree n defined on $x \in [-1,1]$. Let $x_j,\ \ 0 \le j \le n$ and $\gamma_j,\ \ 0 \le j \le n$ be the Gauss-Legendre quadrature points and weights. Finally, let $\ell_j(x),\ \ 0 \le j \le n$ be the Lagrange characteristic interpolating polynomials defined with nodes at the Gauss-Legendre quadrature points.

Show that the following summation by parts formula holds:

$$\sum_{j=0}^{n} U'(x_j)\, V(x_j)\, \gamma_j = (U(1)V(1) - U(-1)V(-1)) - \sum_{j=0}^{n} U(x_j)\, V'(x_j)\, \gamma_j$$

Problem 3.6

Consider the quadrature formula $I_3(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $\mathcal{I}(f) = \int_0^1 f(x)dx$, where $f \in C^4([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that $I_3(f)$ has degree of exactness $s = 2$. Also, for the resulting method, determine the leading term of the quature error, i.e., find C, d and r in $\mathcal{I}(f) - I_3(f) = Ch^r f^{(d)} + O(h^{r+1})$.

Problem 3.7

Consider numerically approximating the integral

$$\mathcal{I}(f) = \int_a^b f(x)dx$$

using the **open Newton-Cotes** with $n = 2$, i.e., 3 points

$$I_2^{(o)} = \frac{4}{3} h_2 \left[2f(x_0) - f(x_1) + 2f(x_2) \right].$$

(3.7.a) Determine C, d, and s in the error expression

$$\mathcal{I} - I_2^{(o)} = C(b-a)^d f^{(s)} + O\left((b-a)^{d+1}\right)$$

(3.7.b) Suppose $\mathcal{I}$ is numerically approximated using a composite method, $I_{c2}^{(o)}$, based on $I_2^{(o)}$ with m intervals each of size $H = (b-a)/m$. Determine C, d, and s in the error expression

$$\mathcal{I} - I_{c2}^{(o)} = C(b-a)H^d f^{(s)} + O\left(H^{d+1}\right)$$

(3.7.c) Suppose global step halving is used to define a coarse grid with m intervals of size H_c and a fine grid with $2m$ intervals of size $H_f = \alpha H_c$ for the composite method, $I_{c2}^{(o)}$, where $\alpha = 0.5$. Determine, per interval on the coarse grid, the number of function evaluations made on the coarse grid that can be reused on the fine grid.

(3.7.d) Determine the number of new function evaluations required per interval on the fine grid to generate $I_{c2}^{(o)}$ on the fine grid.

(3.7.e) Is there a step refinement $\alpha \neq 0.5$ that allows you to reuse all of the function evaluations from the coarse grid with interval size H_c on the fine grid with interval size $H_f = \alpha H_c$?

Problem 3.8

3.8.a

Suppose $y_j = y(x_j)$ for a set of points x_j with constant spacing $h = x_j - x_{j-1}$. Consider the following linear difference formula, D:

$$Dy_i = \frac{1}{h}\left(\alpha_2 y_{i+2} + \alpha_1 y_{i+1} + \alpha_0 y_i\right)$$

Determine the coefficients $\alpha_i, \;\; i = 0,\ 1,\ 2$ to maximize the order to which Dy_i approximates $y'(x_i)$ and determine C and k in the resulting error expression

$$y'(x_i) = Dy_i + Ch^k y^{(k+1)} + O(h^{k+1}).$$

3.8.b

Consider applying the the difference operator to the function $y(x) = \sin x$.

(i) Take $x_i = \pi/4$. What value of h must be used to get an approximation of $y'(\pi/4)$ that satisfies

$$|y'(\pi/4) - Dy_i| \leq 10^{-4}?$$

(ii) Apply the difference operator to $y(x_i)$ with $x_i = \pi/4$ using the h you have derived to verify that your error is less than the required bound.