Study Questions Homework 4 Foundations of Computational Math 2 Spring 2025

Linear Multistep Methods

Problem 4.1

Consider the following linear multistep method:

$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$

4.1.a. Determine, p, the order of consistency of the method.

4.1.b. Determine the coefficient, C_{p+1}, in the discretization error d_n.

4.1.c. Consider the application of the method to $y' = 0$ with $y_0 = 0$ and $y_1 = \epsilon$, i.e., a perturbed initial condition. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Problem 4.2

Consider the following linear multistep method:

$$y_n = y_{n-2} + \frac{h}{3}(f_n + 4f_{n-1} + f_{n-2})$$

The method is 0-stable but it is weakly stable.

4.2.a. Determine the discretization error d_n.

4.2.b. Consider the application of the method to $y' = \lambda y$. Write the recurrence that yields y_n.

4.2.c. Let y_n, $n = 0, 1, \ldots$ be the numerical solution of $y' = \lambda y$ from the previous part of the problem. Show that $|y_n| \to \infty$ as $n \to \infty$, i.e., the numerical method is unstable.

Problem 4.3

Recall, our model problem

$$f = \lambda(y - F(t)) + F'(t) \quad y(0) = y_0$$
$$y(t) = (y_0 - F(0))e^{\lambda t} + F(t)$$

Take $F(t) = \sin t$ and $y(0) = 1$ and consider $y(t)$ on $0 \le t \le 1$.

Consider the methods

- Method 1
$$y_n = -4y_{n-1} + 5y_{n-2} + h(4f_{n-1} + 2f_{n-2})$$
- Method 2 – explicit midpoint
$$y_n = y_{n-2} + 2hf_{n-1}$$
- Method 3 – Adams Bashforth two-step
$$y_n = y_{n-1} + \frac{h}{2}(3f_{n-1} - f_{n-2})$$
- Method 4 – Adams Moulton one-step (Trapezoidal Rule)
$$y_n = y_{n-1} + \frac{h}{2}\left(f_n + f_{n-1}\right)$$
- Method 5 – BDF one-step (Backward Euler)
$$y_n = y_{n-1} + hf_n$$
- Method 6 – BDF two-step
$$y_n = \frac{4}{3}y_{n-1} - \frac{1}{3}y_{n-2} + \frac{2}{3}hf_n$$

Apply the methods to the model problem using **exact initial conditions**, e.g., $y_0 = y(0)$ and $y_1 = y(h)$ for a two-step LMS method. Consider $\lambda = 10$, $\lambda = -10$, and $\lambda = -500$.

1. Use various fixed stepsizes, e.g., $h = 0.01$ and smaller is a good place to start, and apply the methods. Organize your observations on the accuracy and error damping in the transient and quiescent regions for each method and fixed stepsize.

2. Explain your observations based on the local error (h time the local discretization error), the absolute stability properties, and stiff decay (or lack of it).

3. Consider using multiple stepsizes to improve the accuracy and efficiency for some of the methods. You should use at least two stepsizes: one for the transient region and one for the quiescent region. You should also consider using more than one in the quiescent region, i.e., a series of stepsizes that increase from the stepsize used in the transient region to the larger stepsize expected based on the local error of the slowly changing solution to the IVP.

4. Check other values of λ and see if you can predict the behavior of the methods.

Comment on implicit methods: Note that some of these methods are implicit. Due to its simple form, for this model problem you can derive a closed form of the solution of the implicit method's equation defining y_n, i.e., there is no need to use functional iteration or Newton's method to solve a "corrector" equation or to choose a predictor method to get the initial guess of y_n. To see this note that any of the implicit LMS methods can be written as

$$y_n - h\beta_0 f(y_n, t_n) = S$$

where S is a known value that depends upon the method, stepsizes and past points. Substituting f into the expression and solving yields

$$y_n = [S + h\beta_0(F' - \lambda F)]\,(1 - h\lambda\beta_0)^{-1}$$

which can be used to advance the implicit methods to y_n, t_n. (You should verify this solution analytically and by checking that it has a small residual for the implicit method's formula as it is applied during integration.)

Problem 4.4

Recall that to derive the second order BDF with fixed stepsize the interpolating polynomial $p(t)$ was used with uniform separation between t_n, t_{n-1}, and t_{n-2}. Consider the nonconstant stepsize second order BDF where

$$h_n = t_n - t_{n-1}$$
$$h_{n-1} = t_{n-1} - t_{n-2}$$

Starting with the Newton divided difference form of the appropriate interpolating polynomial $\phi(t)$, derive the nonconstant stepsize second order BDF with the form

$$\alpha_0(h_n, h_{n-1})y_n + \alpha_1(h_n, h_{n-1})y_{n-1} + \alpha_2(h_n, h_{n-1})y_{n-2} = h_n f_n$$

where the real coefficients $\alpha_i(h_n, h_{n-1})$ are functions of the two stepsizes h_n and h_{n-1}.

Problem 4.5

Adapt the techniques used to derive the Adams Moulton 2-step method with constant step

$$y_n = y_{n-1} + h\left(\frac{5}{12}f_n + \frac{8}{12}f_{n-1} - \frac{1}{12}f_{n-2}\right).$$

to find the expression for a nonconstant stepsize 2-step Adams Moulton method with stepsizes $h_n = t_n - t_{n-1}$ and $h_{n-1} = t_{n-1} - t_{n-2}$. Give the result in the form:

$$y_n = y_{n-1} + h\left(\beta_0(h_n, h_{n-1})f_n + \beta_1(h_n, h_{n-1})f_{n-1} - \beta_2(h_n, h_{n-1})f_{n-2}\right).$$

where the real coefficients $\beta_i(h_n, h_{n-1})$ are functions of the two stepsizes h_n and h_{n-1}.

Runge Kutta Methods

Problem 4.6

Consider the Runge Kutta method called the explicit trapezoidal rule given by:

$$\hat{y}_1 = y_{n-1} + hf(t_{n-1}, y_{n-1})$$
$$y_n = y_{n-1} + \frac{h}{2}\big(f(t_{n-1}, y_{n-1}) + f(t_n, \hat{y}_1)\big)$$

Show that the method has truncation error $O(h^2)$.

Problem 4.7

Consider the Runge Kutta method called the implicit midpoint rule given by:

$$\hat{y}_1 = y_{n-1} + \frac{h}{2}f_1$$
$$f_1 = f(t_{n-1} + \frac{h}{2}, \hat{y}_1)$$
$$y_n = y_{n-1} + hf_1$$

An alternate form of the the method is given by:

$$y_n = y_{n-1} + hf\big(\frac{t_n + t_{n-1}}{2}, \frac{y_n + y_{n-1}}{2}\big)$$

Show that the two forms are identical.

Problem 4.8

Consider the Runge Kutta method called the implicit midpoint rule given by:

$$\hat{y}_1 = y_{n-1} + \frac{h}{2}f_1$$
$$f_1 = f(t_{n-1} + \frac{h}{2}, \hat{y}_1)$$
$$y_n = y_{n-1} + hf_1$$
$$\left(\begin{array}{c|c} \gamma_1 & \alpha_{11} \\ \hline & \beta_1 \end{array}\right) = \left(\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}\right)$$

Show that the method has truncation error $O(h^2)$.

Problem 4.9

Consider the general form of a 2-stage Explicit RK method:

$$\hat{y}_1 = y_{n-1}, \quad f_1 = f(t_{n-1}, \hat{y}_1)$$

$$\hat{y}_2 = y_{n-1} + \alpha_{21} f_1, \quad f_2 = f(t_{n-1} + \gamma_2 h, \hat{y}_2)$$

$$y_n = y_{n-1} + h\left(\beta_1 f_1 + \beta_2 f_2\right)$$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & 0 & 0 \\ \gamma_2 & \alpha_{21} & 0 \\ \hline & \beta_1 & \beta_2 \end{array}$$

$$\gamma_2 = \alpha_{21}$$

4.9.a. Determine the set of equations that the free parameters must satisfy in order to achieve method with order 2.

4.9.b. Is there a single such method? If so prove it. If not discuss the number of free parameters and give examples of methods and potential parameterized tables that define families of methods.

Problem 4.10

Consider the 4 stage classical RK4 method with order 4. Derive the function $R(z)$ that determines the region of absolute stability region of the method by the equation

$$y_n = R(z) y_{n-1}$$

where $z = h\lambda$.

Advanced Problems

Problem 4.11

Recall, we have examined the polynomials $\rho(\xi)$ and $\sigma(\xi)$ associated with a linear multistep method. $\rho(\xi)$ is related to the analysis of strong, weak and 0-stability of the method and $\mu(\xi) = \rho(\xi) - h\lambda\sigma(\xi)$ is used to determine the absolute stabililty properties and region of the method. All three parts of this question relate in some way to these three polynomials.

4.11.a

i. What stability properties of the method can be examined by looking at the roots of $\sigma(\xi)$? (Take care when $\sigma(\xi)$ has lower degree than $\rho(\xi)$.)

ii. Explain the statement "The Adams methods are as strongly stable as any linear multistep method can possibly be."

iii. The motivation for the design of the BDF methods is stiff decay. Explain how the form of the BDFs is linked to this motivation.

4.11.b

Consider the linear multistep method:

$$y_n - y_{n-2} = 2hf_{n-1}$$

Discuss the absolute stability properties of the method. You may do this by determining the boundary of the absolute stability region or by other means.

4.11.c

Consider the linear multistep method:

$$y_n = y_{n-1} + h\left(\frac{9}{16}f_n + \frac{6}{16}f_{n-1} + \frac{1}{16}f_{n-2}\right)$$

(i) Is the method convergent?

(ii) The method is **not** an Adams Moulton method. Examine the absolute stability properties of this method and identify the main advantage or disadvantages this method compared to the 2-step Adams Moulton method. You do not have to determine the entire boundary to solve this problem.

Problem 4.12

Assume you have an implicit k step linear multistep method of the form

$$y_n = h\beta_0 f_n + \sum_{i=1}^{k} (h\beta_i f_{n-i} - \alpha_i y_{n-i}) = h\beta_0 f_n + S_*$$

that has order p, i.e.,

$$y(t_n) = h\beta_0 f(y(t_n)) + \sum_{i=1}^{k} (h\beta_i f(y(t_{n-i})) - \alpha_i y(t_{n-i})) + O(h^{p+1})$$

where the t argument to f has been suppressed for convenience.

Suppose you apply a $P(EC)^m E$ method to solve approximately this implicit equation to determine y_n and the predictor is assumed to have order $\ell < p$ accuracy, i.e.,

$$y(t_n) = y_n^{[0]} + O(h^{\ell+1})$$

$$y_n^{[j]} = h\beta_0 f(y_n^{[j-1]}) + S_*$$

(4.12.a) Assume that $y_{n-i} = y(t_{n-i})$ for $i = 1, \ldots, k$ and show that each iteration of the EC step increases the order of accuracy of $y_n^{[j]}$ by 1, i.e.,

$$y(t_n) = y_n^{[j]} + O(h^{l+1+j}) + O(h^{p+1})$$

(4.12.b) What order ℓ for the predictor would you recommend be used in practice and why?

Problem 4.13

4.13.a

Solutions to ODE initial value problems often satisfy invariants, i.e., a condition on $y(t)$ that is true for all t in the interval defined by the problem. For example, the solution $y(t) \in \mathbb{R}^m$ to

$$y' = f(y,t), \quad y(t_0) = y_0$$

where $f(y,t) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ is Lipchitz in y could have a constant size as measured by the vector 2-norm in $\mathbb{R}^m$, i.e.,

$$\|y_0\|_2^2 = y_0^T y_0 = \|y(t)\|_2^2 = y(t)^T y(t)$$

for all t in the interval of the problem.

What condition must hold for $y(t)$ and $f(y,t)$ to give a solution that is invariant in the vector 2-norm in $\mathbb{R}^m$? Justify your answer.

4.13.b

If a solution to an IVP satisfies an invariant it is of interest to know which numerical methods preserve that invariant in the numerical solution (assuming exact arithmetic, i.e., no roundoff).

The system of two ODEs

$$Y'(t) = MY(t), \quad Y(0) = Y_0$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where $\omega > 0$ has a solution for which $\|Y(t)\|_2^2 = \|Y_0\|_2^2$ where $Y(t) \in \mathbb{R}^2$ contains $y_1(t)$ and $y_2(t)$ as its components and $M \in \mathbb{R}^{2\times 2}$. In fact, for any $Y_0 \in \mathbb{R}^2$ the solution stays on the circle with radius $\|Y_0\|_2^2$.

Derive expressions for the numerical solution Y_n and $\|Y_n\|_2^2$ that results from applying the Trapezoidal Rule to the problem in part above and use them to determine if the Trapezoidal Rule preserves $\|Y_n\|_2^2 = \|Y_0\|_2^2$?