Study Problems 1 Numerical Linear Algebra 1 Spring 2024

Problem 1.1

Consider the vector space $\mathbb{R}^4$

1.1.a. Specify a subspace of $\mathbb{R}^4$ with dimension 2 by giving a basis for the subspace.

1.1.b. Show that the basis for a subspace is not unique by giving another basis for the same subspace given in (1.1.a).

Problem 1.2

This problem considers three basic vector norms: $\|.\|_1$,$\|.\|_2$,$\|.\|_\infty$.

1.2.a. Prove that $\|.\|_1$ is a vector norm.

1.2.b. Prove that $\|.\|_\infty$ is a vector norm.

1.2.c. Consider $\|.\|_2$.

(i) Show that $\|.\|_2$ is definite.

(ii) Show that $\|.\|_2$ is homogeneous.

(iii) Show that for $\|.\|_2$ the triangle inequality follows from the Cauchy inequality $|x^H y| \leq \|x\|_2 \|y\|_2$.

(iv) Assume you have two vectors x and y such that $\|x\|_2 = \|y\|_2 = 1$ and $x^H y = |x^H y|$, prove the Cauchy inequality holds for x and y.

(v) Assume you have two arbitrary vectors $\tilde{x}$ and $\tilde{y}$. Show that there exists x and y that satisfy the conditions of part (iv) and $\tilde{x} = \alpha x$ and $\tilde{y} = \beta y$ where α and β are scalars.

(vi) Show the Cauchy inequality holds for two arbitrary vectors $\tilde{x}$ and $\tilde{y}$.

Problem 1.3

Let $y \in \mathbb{R}^m$ and $\|y\|$ be any vector norm defined on $\mathbb{R}^m$. Let $x \in \mathbb{R}^n$ and A be an $m \times n$ matrix with $m > n$.

1.3.a. Show that the function $f(x) = \|Ax\|$ is a vector norm on $\mathbb{R}^n$ if and only if A has full column rank, i.e., $rank(A) = n$.

1.3.b. Suppose we choose $f(x)$ from part (1.3.a) to be $f(x) = \|Ax\|_2$. What condition on A guarantees that $f(x) = \|x\|_2$ for any vector $x \in \mathbb{R}^n$?

Problem 1.4

1.4.a. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be nonsingular matrices. Show $(AB)^{-1} = B^{-1}A^{-1}$.

1.4.b. Suppose $A \in \mathbb{R}^{m\times n}$ with $m > n$ and let $M \in \mathbb{R}^{n\times n}$ be a nonsingular square matrix. Show that $\mathcal{R}(A) = \mathcal{R}(AM)$ where $\mathcal{R}()$ denotes the range of a matrix.

Problem 1.5

Consider the matrix

$$L = \begin{pmatrix} \lambda_{11} & 0 & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & \lambda_{44} \end{pmatrix}$$

Suppose that $\lambda_{11} \neq 0$, $\lambda_{33} \neq 0$, $\lambda_{44} \neq 0$ but $\lambda_{22} = 0$.

1.5.a. Show that L is singular.

1.5.b. Determine a basis for the nullspace $\mathcal{N}(L)$.

Problem 1.6

Suppose that $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ and let $E = uv^T$.

1.6.a. Show that $\|E\|_F = \|E\|_2 = \|u\|_2\|v\|_2$.

1.6.b. Show that $\|E\|_\infty = \|u\|_\infty\|v\|_1$.

Problem 1.7

Show that for any vector norm on $\mathbb{C}$,

$$\forall\, x,\ y \in \mathbb{C} \quad \|x - y\| \geq |\ \|x\| - \|y\|\ |$$

Problem 1.8

Let $\mathcal{S}_1 \subset \mathbb{R}^n$ and $\mathcal{S}_2 \subset \mathbb{R}^n$ be two subspaces of $\mathbb{R}^n$.

1.8.a. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Show that x_1 and x_2 are linearly independent.

1.8.b. Suppose $x_1 \in \mathcal{S}_1$, $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. $x_2 \in \mathcal{S}_2$, and $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$. Also, suppose that $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $x_3 \neq 0$, i.e., the intersection is not empty. Show that x_1, x_2 and x_3 are linearly independent.

Problem 1.9

Suppose $A \in \mathbb{R}^{m \times n}$, $m \geq n$ and $rank(A) = p \leq n$. Show that there exists $X \in \mathbb{R}^{m \times p}$ and $Y \in \mathbb{R}^{\times p}$ such that $rank(X) = rank(Y) = p$ and

$$A = XY^T$$

Problem 1.10

Let $n = 4$ and consider the lower triangular system $Lx = f$ of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda_{21} & 1 & 0 & 0 \\ \lambda_{31} & \lambda_{32} & 1 & 0 \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Recall, that the column-oriented algorithm can be derived from a factorization $L = L_1 L_2 L_3$ where L_i was an elementary unit lower triangular matrix associated with the i-th column of L.

Show that the row-oriented algorithm can be derived from a factorization of L of the form

$$L = R_2 R_3 R_4$$

where R_i is associated with the i-th row of L.

Problem 1.11

Recall that any unit lower triangular matrix $L \in \mathbb{R}^{n \times n}$ can be written in factored form as

$$L = M_1 M_2 \cdots M_{n-1} \tag{1}$$

where $M_i = I + l_i e_i^T$ is an elementary unit lower triangular matrix (column form). Given the ordering of the elementary matrices, this factorization did not require any computation.

Consider a simpler elementary unit lower triangular matrix (element form) that differs from the identity in **one off-diagonal element** in the strict lower triangular part, i.e.,

$$E_{ij} = I + \lambda_{ij} e_i e_j^T$$

where $i \neq j$.

1.11.a. Show that computing the product of two element form elementary matrices is simply superposition of the elements into the product given by

$$E_{ij} E_{rs} = I + \lambda_{ij} e_i e_j^T + \lambda_{rs} e_r e_s^T$$

whenever $j \neq r$.

1.11.b. Show that if $j \neq r$ and $i \neq s$ then computing $E_{ij} E_{rs}$ with requires no computation and

$$E_{ij} E_{rs} = E_{rs} E_{ij}$$

i.e., the matrices commute.

1.11.c. Write a column form elementary matrix M_i in terms of element form elementary matrices. Does the order of the E_{ji} matter in this product?

1.11.d. Show how it follows that the factorization of (1) is easily expressed in terms of element form elementary matrices.

1.11.e. Show that the expression from part (1.11.d) can be rearranged to form $L = R_2 \dots R_n$ where $R_i = I + e_i r_i^T$ is an elementary unit lower triangular matrix in row form.

Problem 1.12

Consider the matrix-vector product $x = Lb$ where L is an $n \times n$ unit lower triangular matrix with **all** of its nonzero elements equal to 1. For example, if $n = 4$ then

$$x = Lb$$
$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

The vector x is called the scan of b. Show that, given the vector b, the vector x can be computed in $O(n)$ computations rather than the $O(n^2)$ typically required by a matrix vector product. Express your solution in terms of matrices and vectors.

Problem 1.13

We have the following theorem relating inner products and norms.

Theorem 1. *Let $\mathcal{V}$ be a real vector space with a norm $\|v\|$.*

1. *If the norm $\|v\|$ satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{2}$$

   for every pair of vectors $x \in \mathcal{V}$ and $y \in \mathcal{V}$ then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

   is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

2. *If $\|v\|$ does not satisfy the parallelogram law (2) for every pair of vectors $x \in \mathcal{V}$ and $y \in \mathcal{V}$ then it is* **not** *generated by an inner product.*

Consider $\mathcal{V} = \mathbb{R}^2$.

1. Show that the vector p-norm with $p = 1$, $\|v\|_1$ is not generated by an inner product.
2. Show that the vector p-norm with $p = 3$, $\|v\|_3$ is not generated by an inner product.
3. Does this imply that the these two vector norms are not generated by an inner product for any $\mathcal{V} = \mathbb{R}^n$?

Problem 1.14

Theorem 2. *If $\mathcal{V}$ is a real vector space with a norm $\|v\|$ that satisfies the parallelogram law*

$$\forall x, \quad y \in \mathcal{V}, \quad \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \tag{3}$$

then the function

$$f(x,y) = \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$$

is an inner product on $\mathcal{V}$ and $f(x,x) = \|x\|^2$.

This problem proves this theorem by a series of lemmas. Prove each of the following lemmas and then prove the theorem.

Lemma 3. $\forall x \in \mathcal{V}$

$$f(x,x) = \|x\|^2$$

Lemma 4. $\forall x, \quad y \in \mathcal{V}$ *$f(x,x)$ is definite and $f(x,y) = f(y,x)$, i.e., (f is symmetric)*

Lemma 5. *The following two "cosine laws" hold* $\forall x, \ \ y \in \mathcal{V}$*:*

$$2f(x,y) = \|x+y\|^2 - \|x\|^2 - \|y\|^2 \tag{4}$$

$$2f(x,y) = -\|x-y\|^2 + \|x\|^2 + \|y\|^2 \tag{5}$$

Lemma 6. $\forall x, \ \ y \in \mathcal{V}$*:*

$$|f(x,y)| \leq \|x\|\|y\| \tag{6}$$

$$f(x,y) = \gamma\|x\|\|y\|, \quad sign(\gamma) = sign(f(x,y)), \quad 0 \leq |\gamma| \leq 1 \tag{7}$$

Lemma 7. $\forall x, \ \ y, \ \ z \in \mathcal{V}$*:*

$$f(x+z,y) = f(x,y) + f(z,y)$$

Lemma 8. $\forall x, \ \ y \in \mathcal{V}, \ \ \alpha \in \mathbb{R}$

$$f(\alpha x, y) = \alpha f(x,y)$$

Problem 1.15

Definition 0.1. $M \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix if $M = M^T$ and for any $x \in \mathbb{R}^n$ M satisfies

$$x^T Mx \geq 0.$$

M is symmetric positive definite matrix if for any $x \neq 0 \in \mathbb{R}^n$

$$x^T Mx > 0$$

with equality only if $x = 0$.

1.15.a. Show that any eigenvalue λ of a symmetric positive semidefinite satisfies $\lambda \geq 0$.

1.15.b. Show that any eigenvalue λ of a symmetric positive definite satisfies $\lambda > 0$.

1.15.c. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ be a given matrix. Show that $A^T A \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix.

1.15.d. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ be a given matrix. Give a condition on A that guarantees that $A^T A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.