Study Problems 2 Numerical Linear Algebra 1 Spring 2024

Problem 2.1

Suppose $A \in \mathbb{C}^{n\times n}$ (not necessarily Hermitian) and show that

$$A = A_D + A_N$$

where A_D is a diagonalizable matrix, i.e., nondefective, A_N is a nilpotent matrix and $A_D A_N = A_N A_D$.

(Recall that a nilpotent $n \times n$ matrix B is such that $B^k = 0$ for some $k \le n$.)

Problem 2.2

Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be normal matrices that are **not diagonal matrices**. Give a sufficient condition for the matrix product of A and B to commute, i.e., $AB = BA$.

Problem 2.3 (25 points)

2.3.a. Suppose (v_1, λ_1) and (v_2, λ_2) are eigenpairs for a matrix $A \in \mathbb{C}^{n\times n}$. Show that if $\lambda_1 \neq \lambda_2$ then v_1 and v_2 are linearly independent.

2.3.b. Suppose that (v_i, λ_i) for $i = 1, \ldots, n$ are eigenpairs for a matrix $A \in \mathbb{C}^{n\times n}$ where $\lambda_i \neq \lambda_j$ for any $i \neq j$. Show that $\{v_i\}$ for $i = 1, \ldots, n$ is a set of linear independent vectors.

2.3.c. Define the matrix $V \in \mathbb{C}^{n\times n}$ by letting its i=th column be the eigenvector v_i given in part (2.3.b) for n distinct eigenvalues, i.e.

$$V = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix}.$$

What decomposition can be written for the matrix A using V and the n distinct eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$.

2.3.d. Suppose the matrix $Q \in \mathbb{C}^{n\times n}$ is a unitary matrix, i.e, its columns are such that $\|Qe_i\|_2 = 1$ and $q_i^H q_j = 0$ when $i \neq j$. If $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$ are an eigenvalue and eigenvector of Q, what can you say about $|\lambda|$?

Problem 2.4

A matrix $A \in \mathbb{C}^{n\times n}$ is nilpotent if $A^k = 0$ for some integer $k > 0$. Prove that the only eigenvalue of a nilpotent matrix is 0.

Problem 2.5

(Modified Problem 7.1.1 Golub and Van Loan (3rd Ed.) p. 318, Problem 7.1.1(a) Golub and Van Loan (4th Ed.) p. 355)

2.5.a. Suppose $U \in \mathbb{C}^{n\times n}$ is block upper triangular of the form

$$U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}, \quad U_{11} \in \mathbb{C}^{k\times k}, \quad U_{22} \in \mathbb{C}^{n-k\times n-k}.$$

Show that if U is normal then U is a block diagonal matrix, i.e., $U_{12} = 0$, and the diagonal blocks U_{11} and U_{22} are both normal matrices.

2.5.b. Show that if $U \in \mathbb{C}^{n\times n}$ is upper triangular and normal then it is a diagonal matrix.

2.5.c. Show that if $L \in \mathbb{C}^{n\times n}$ is lower triangular and normal then it is a diagonal matrix. (Do this without repeating the entire proof above.)

Problem 2.6

(Problem 7.1.1(c) Golub and Van Loan (4th Ed.) p. 355)

Prove that a matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix.

Problem 2.7

Let the matrix $A \in \mathbb{C}^{n\times n}$ be unitary. Show that if λ is an eigenvalue of A then $|\lambda| = 1$.

Problem 2.8

2.8.a. Suppose $T \in \mathbb{C}^{n\times n}$ is an upper triangular matrix with its diagonal elements ordered so that the m diagonal block submatrices $T_{jj} \in \mathbb{C}^{k_j\times k_j}$, $j = 1, \ldots, m$ have a distinct value μ_j in all of its diagonal positions, i.e., $T_{jj} = \mu_j I_{k_j} + N_j$

where N_j is strictly upper triangular for $k = 1, \ldots, k_j$ and $\mu_j \neq \mu_{\tilde{j}}$ when $j \neq \tilde{j}$. Show that there exists a nonsingular matrix $M \in \mathbb{C}^{n\times n}$ such that $M^{-1}TM$ is a block diagonal matrix with diagonal blocks T_{jj}.

Hint: Show that there exists nonsingular $M_1 \in \mathbb{C}^{n\times n}$ such that

$$M_1^{-1}TM_1 = M_1^{-1}\begin{pmatrix} T_{11} & Y_1^H \\ 0 & R_1 \end{pmatrix} M_1 = \begin{pmatrix} T_{11} & 0^H \\ 0 & R_1 \end{pmatrix}$$

where $T_{11} \in \mathbb{C}^{k_1\times k_1}$ as defined above and $R_1 \in \mathbb{C}^{(n-k_1)\times(n-k_1)}$ is upper triangular with diagonal blocks T_{jj}, $j = 2, \ldots m$, that are upper triangular as described above and $Y_1^H \in \mathbb{C}^{k_1\times(n-k_1)}$ contains the elements of the first k_1 rows not in the triangular diagonal block T_{11}. Then construct M from a series of these transformations.

2.8.b. Show that any $A \in \mathbb{C}^{n\times n}$ can be transformed, using a similarity tranformation, to block a diagonal matrix where the diagonal blocks are upper triangular, each diagonal block can be written $T_{jj} = \mu_j I_{k_j} + N_j$ where N_j is strictly upper triangular and $\mu_j \neq \mu_{\tilde{j}}$ when $j \neq \tilde{j}$.

Problem 2.9

Consider computing the matrix vector product $y = Tx$, i.e., you are given T and x and you want to compute y. Suppose further that the matrix $T \in \mathbb{R}^{n\times n}$ is tridiagonal with constant values on each diagonal. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}$$

(2.9.a) Write a simple loop-based psuedo-code that computes $y = Tx$ for such a matrix $T \in \mathbb{R}^n$.

(2.9.b) How many operations are required as a function of n?

(2.9.c) Describe your data structures and determine how many storage locations are required as a function of n?

(2.9.d) Suppose the diagonals of T are not constant. For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}.$$

Modify your algorithm to handle non-constant diagonal form and discuss the modifications made to the data structures required by the modification.

Problem 2.10

Suppose you are given a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and consider the computation of the matrix-vector product $v \leftarrow Au$ where $u \in \mathbb{R}^n$ is given and $v \in \mathbb{R}^n$ is computed.

2.10.a Since the matrix is symmetric, there are only $n(n+1)/2$ elements that are free to choose while the others are set due to symmetry. Describe a data structure that would only store $n(n+1)/2$ values that specify A.

2.10.b Give the mappings from mathematical quantities such as $\alpha_{ij} = e_i^T A e_j$, $\nu_i = e_i^T v$, and $\mu_i = e_i^T u$ to the data structures you use for A, v, and u.

2.10.c Describe an algorithm using pseudo-code that uses your data structure to implement the computation of the matrix-vector product, $Au \rightarrow v$, given A and u. Make sure you point out all of the relevant features that influence efficiency.

When describing the algorithm use Matlab coding forms as shown in the posted solution to Program 1.

Problem 2.11

Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular and that $A = LU$ is its LU factorization. Give an algorithm that can compute, $e_i^T A^{-1} e_j$, i.e., the (i, j) element of A^{-1} in approximately $(n - j)^2 + (n - i)^2$ operations.

Problem 2.12

Consider an $n \times n$ real matrix where

- $\alpha_{ij} = e_i^T A e_j = -1$ when $i > j$, i.e., all elements strictly below the diagonal are -1;

- $\alpha_{ii} = e_i^T A e_i = 1$, i.e., all elements on the diagonal are 1;
- $\alpha_{in} = e_i^T A e_n = 1$, i.e., all elements in the last column of the matrix are 1;
- all other elements are 0

For $n = 4$ we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

2.12.a. Compute the factorization $A = LU$ for $n = 4$ where L is unit lower triangular and U is upper triangular.

2.12.b. What is the pattern of element values in L and U for any n?

Problem 2.13

Suppose you have the LU factorization of an $i \times i$ matrix $A_i = L_i U_i$ and suppose the matrix A_{i+1} is an $i+1 \times i+1$ matrix formed by adding a row and column to A_i, i.e.,

$$A_{i+1} = \begin{pmatrix} A_i & a_{i+1} \\ b_{i+1}^T & \alpha_{i+1,i+1} \end{pmatrix}$$

where a_{i+1} and b_{i+1} are vectors in $\mathbb{R}^i$ and $\alpha_{i+1,i+1}$ is a scalar.

2.13.a. Derive an algorithm that, given L_i, U_i and the new row and column information, computes the LU factorization of A_{i+1} **and identify the conditions under which the step will fail**.

2.13.b. What computational primitives are involved?

2.13.c. Show how this basic step could be used to form an algorithm that computes the LU factorization of an $n \times n$ matrix A.

Problem 2.14

Suppose you are computing a factorization of the $A \in \mathbb{C}^{n \times n}$ with partial pivoting and at the beginning of step i of the algorithm you encounter the the transformed matrix with the form

$$TA = A^{(i-1)} = \begin{pmatrix} U_{11} & U_{12} \\ 0 & A_{i-1} \end{pmatrix}$$

where $U_{11} \in \mathbb{R}^{i-1 \times i-1}$ and nonsingular, and $U_{12} \in \mathbb{R}^{i-1 \times n-i+1}$ contain the first $i-1$ rows of U. Show that if the first column of A_{i-1} is all 0 then A must be a singular matrix.

Problem 2.15

Suppose you are given a tridiagonal matrix $T \in \mathbb{R}^{n \times n}$ For example, if $n = 6$ then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}.$$

Assume that $T \in \mathbb{R}^{n \times n}$ is strictly diagonally dominant by columns and that the elements on the main diagonal are all positive, i.e., $\alpha_i > 0$ for $1 \leq i \leq n$.

(2.15.a) Determine the form of the matrix after one step of LU factorization without pivoting (which is not needed since diagonal dominance is assumed).

(2.15.b) How many operations were required to perform the single step?

(2.15.c) What is the structure of the active part of the matrix after one step, i.e., what is the Schur complement of T with respect to α_1?

(2.15.d) Suppose in addition it is assumed that $\beta_i \geq 0$ for $1 \leq i \leq n-1$ and $\gamma_i \geq 0$ for $2 \leq i \leq n$. What is the maximum growth of the elements in the active part of the matrix after one step relative to the elements in T?

(2.15.e) What does this one step growth imply about the Wilkinson growth factor after completing the LU factorization?

(2.15.f) Does your conclusion about the growth factor change if the assumptions on the off-diagonal elements are changed to $\beta_i \leq 0$ for $1 \leq i \leq n-1$ and $\gamma_i \geq 0$ for $2 \leq i \leq n$?

Problem 2.16

Suppose $A \in \mathbb{R}^{n \times n}$ is a nonsymmetric nonsingular diagonally dominant matrix with the following nonzero pattern (shown for $n = 6$)

$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & 0 & 0 & 0 & 0 \\ * & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

It is known that a diagonally dominant (row or column dominant) matrix has an LU factorization and that pivoting is not required for numerical reliability.

2.16.a. Describe an algorithm that solves $Ax = b$ as efficiently as possible.

2.16.b. Given that the number of operations in the algorithm is of the form $Cn^k + O(n^{k-1})$, where C is a constant independent of n and $k > 0$, what are C and k?

Problem 2.17

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix, with A and A^{-1} partitioned as follows

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$$

where $A_{11} \in \mathbb{R}^{k \times k}$ and $\tilde{A}_{11} \in \mathbb{R}^{k \times k}$.

2.17.a. Assume A_{11}^{-1} and A_{22}^{-1} exist. Let $S_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ be the Schur complement of A with respect to A_{11} and let $S_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ be the Schur complement of A with respect to A_{22} Show that

$$\tilde{A}_{11} = S_{22}^{-1} \quad \text{and} \quad \tilde{A}_{22} = S_{11}^{-1}.$$

2.17.b. The assumption of the existence of A^{-1} can be turned into a consequence of the existence of the Schur complement. Show that if, $S_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$, the Schur complement of A with respect to A_{11} exists then A is nonsingular if and only if S_{11} is nonsingular. (A similar result can be stated for S_{22}.)

Problem 2.18

Suppose an LU decomposition of a matrix $A \in \mathbb{R}^{n \times n}$ is to be computed with some form of pivoting to ensure existence. Suppose further that the matrix A is made available one row at a time.

(2.18.a) Describe an algorithm such that when the i-th row of A is received the algorithm computes the i-th row of L and the i-th row of U as well as an elementary permutation matrix P_i that ensures existence (and enhances stability).

(2.18.b) What primitives are used on each step of the algorithm and what are the dimensions of the matrices and vectors involved?

(2.18.c) Why does the pivoting strategy in the algorithm guarantee existence?

(2.18.d) What form of decomposition is computed given the pivoting strategy? (Recall, partial pivoting of rows yields $P_R A = LU$, complete pivoting yields $P_R A P_C = LU$, where P_R and P_C are permutations of rows and columns respectively. Characterize the decomposition produced by the algorithm in a similar manner.)

Problem 2.19

(Restated Golub and Van Loan 3rd Ed. p. 103 Problem P3.2.5.)

Define the elementary matrix $N_k^{-1} = I - y_k e_k^T \in \mathbb{R}^{n\times n}$, where $1 \le k \le n$ is an integer, $y_k \in \mathbb{R}^n$ and $e_k \in \mathbb{R}^n$ is the k-th standard basis vector. N_k^{-1} is a Gauss-Jordan transform if it is defined by requiring $N_k^{-1} v = e_k \, \nu_k$ for a particular given vector $v \in \mathbb{R}^n$ whose elements are denoted $\nu_j = e_j^T v$. For example, if $n = 6$ and $k = 3$ then

$$N_3^{-1} = \begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 \\ 0 & 1 & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & * & 1 & 0 & 0 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \nu_5 \\ \nu_6 \end{pmatrix}$$

where $*$ indicates a value that must be determined.

(2.19.a) Determine how to choose y_k and define N_k^{-1} given a vector $v \in \mathbb{R}^n$, i.e., determine the values of the elements of y_k in terms of the values of the elements of v so that $N_k^{-1} v = e_k \, \nu_k$.

(2.19.b) Determine when N_k^{-1} exists and is nonsingular.

(2.19.c) Show how a series of N_k^{-1} can be used to transform a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ into a nonsingular diagonal matrix $D \in \mathbb{R}^{n\times n}$, i.e., all of the off-diagonal elements of D are 0 and all of the diagonal elements are nonzero. You may assume that A is such that all of the N_k^{-1} exist.

(2.19.d) Does the factorization that this transformation induces have any structure other than that in D?

Problem 2.20

It is known that if partial or complete pivoting is used to compute $PA = LU$ or $PAQ = LU$ of a nonsingular matrix then the elements of L are less than 1 in magnitude, i.e., $|\lambda_{ij}| \le 1$. Now suppose $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, i.e., $A = A^T$ and $x \ne 0 \rightarrow x^T A x > 0$. It is known that A has a factorization $A = LL^T$ where L is lower triangular with positive elements on the main diagonal (the Cholesky factorization). Does this imply that $|\lambda_{ij}| \le 1$? If so prove it and if not give an $n \times n$ symmetric positive definite matrix with $n > 3$ that is a counterexample and justify that it is indeed a counterexample.

Problem 2.21

Suppose $PAQ = LU$ is computed via Gaussian elimination with complete pivoting. Show that there is no element in $e_i^T U$, i.e., row i of U, whose magnitude is larger than $|\mu_{ii}| = |e_i^T U e_i|$, i.e., the magnitude of the (i,i) diagonal element of U.

Problem 2.22

Consider $S \in \mathbb{R}^{n \times n}$ whose nonzero elements have the following pattern for $n = 8$:

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & \mu_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \mu_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \delta_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta_3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The pattern generalizes to any n easily. Assume that for any n, S is a nonsingular matrix.

2.22.a We have considered several basic transformations (Gauss transforms, Gauss-Jordan transforms, elementary permutations, Householder reflectors) that can be used to compute factorizations efficiently. Assume that S is diagonally dominant (both row-wise and column-wise).

Using whatever combination of these transformations you think appropriate, describe an algorithm to compute stably a factorization of S for any n that can be used to solve $Sx = b$. **Your algorithm should be designed to require as few computations as possible.** Your solution must include a description of how you exploit the structure of the matrix and its factors.

2.22.b Assume that you have the factorization of S defined by your algorithm from Part (2.22.a), describe an algortihm to solve $Sx = b$. **Your algorithm should be designed to require as few computations as possible.** Your solution must include a description of how you exploit the structure of the matrix and its factors.

2.22.c Determine the order of computational complexity, i.e., give k in $O(n^k)$, when your factorization algorithm is applied to a matrix of any dimension n.

2.22.d Determine the order of computational complexity, i.e., give k in $O(n^k)$, when your algorithm to solve $Sx = b$ given the factorization is applied to a matrix of any dimension n.