Study Problems 3 Numerical Linear Algebra 1 Spring 2024

Problem 3.1

Let x and y be two vectors in $\mathbb{R}^n$.

3.1.a. Show that given x and y the value of $\|x - \alpha y\|_2$ is minimized when

$$\alpha_{min} = \frac{x^T y}{y^T y}$$

3.1.b. Show that $x = y\alpha_{min} + z$ where $y^T z = 0$, i.e., x is easily written as the sum of two orthogonal vectors with specifed minimization properties.

Problem 3.2

Recall that an elementary reflector has the form $Q = I + \alpha z z^T \in \mathbb{R}^{n \times n}$ with $\|z\|_2 \neq 0$.

3.2.a. Show that Q is orthogonal if and only if

$$\alpha = \frac{-2}{z^T z} \quad \text{or} \quad \alpha = 0$$

3.2.b. Given $v \in \mathbb{R}^n$, let $\gamma = \pm\|v\|$ and $z = v + \gamma e_1$. Assuming that $z \neq v$ show that

$$\frac{z^T z}{z^T v} = 2$$

3.2.c. Using the definitions and results above show that $Qv = -\gamma e_1$

Problem 3.3

Let $A \in \mathbb{R}^{n \times k}$ have full column rank. Describe an efficient algorithm based on Householder reflectors, H_i, $1 \leq i \leq k$ that computes a matrix $Q \in \mathbb{R}^{n \times k}$ with orthonormal columns such that

$$\mathcal{R}(A) = \mathcal{R}(Q)$$

i.e., A and Q have the same range space.

Problem 3.4

Consider a Householder reflector, H, in $\mathbb{R}^2$. Show that

$$H = \begin{pmatrix} -\cos(\phi) & -\sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

where ϕ is some angle.

Problem 3.5

Let $A \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times k}$ be given with $Q^TQ = I_k$, i.e., Q has orthonormal columns. Find $M_{min} \in \mathbb{R}^{k\times k}$ such that

$$M_{min} = \underset{M\in\mathbb{R}^{k\times k}}{\operatorname{argmin}} \|AQ - QM\|_F$$

Problem 3.6

3.6.a. Let $H = I + \alpha xx^T \in \mathbb{R}^{n\times n}$, where $\alpha = -2/\|x\|_2^2$, be a Householder reflector. Determine two distinct eigenvalues for H and associated eigenvectors.

3.6.b. Let $\gamma = \cos\theta$ and $\sigma = \sin\theta$ and consider the 2×2 matrix

$$M = \begin{pmatrix} \gamma & \sigma \\ -\sigma & \gamma \end{pmatrix}$$

Determine the eigenvalues and eigenvectors of M.

Problem 3.7

If $A \in \mathbb{C}^{n_1\times n_2}$ and $B \in \mathbb{C}^{n_3\times n_4}$ then the Kronecker product

$$M = A \otimes B \in \mathbb{C}^{n_1 n_3\times n_2 n_4}$$

is defined in terms of blocks $M_{ij} \in \mathbb{C}^{n_3\times n_4}$ for $1 \le i \le n_1$ and $1 \le j \le n_2$ where

$$M_{ij} = \alpha_{ij} B.$$

The Kronecker product is useful for expressing many structured matrix expressions, e.g., the Cooley-Tukey FFT/IFFT.

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $x \in \mathbb{C}^{mn}$, and $y \in \mathbb{C}^{mn}$.

3.7.a. Describe an algorithm to evaluate the matrix vector product

$$y = (A \otimes B)x$$

i.e., given A, B, x determine y.

3.7.b. What is the complexity of the algorithm?

3.7.c. How does the complexity of the algorithm compare to the standard matrix-vector product computation, $y = Mx$, that ignores the structure of M.

Problem 3.8

If $A \in \mathbb{C}^{n_1 \times n_2}$ and $B \in \mathbb{C}^{n_3 \times n_4}$ then the Kronecker product

$$M = A \otimes B \in \mathbb{C}^{n_1 n_3 \times n_2 n_4}$$

is defined in terms of blocks $M_{ij} \in \mathbb{C}^{n_3 \times n_4}$ for $1 \le i \le n_1$ and $1 \le j \le n_2$ where

$$M_{ij} = \alpha_{ij} B.$$

Let $A, B, C, D \in \mathbb{C}^{n \times n}$ be given square matrices.

3.8.a. Show that

$$(I \otimes A)(I \otimes B) = (I \otimes AB)$$

3.8.b. Show that

$$(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I) = (A \otimes B)$$

3.8.c. Show that

$$(A \otimes I)(B \otimes I) = (AB \otimes I)$$

3.8.d. Show that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

3.8.e. Show that if $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ have inverses A^{-1} and B^{-1}, respectively, then $(A \otimes B)$ has an inverse.

Problem 3.9

Let $A \in \mathbb{R}^{n \times k}$ have rank k, i.e., have k linearly independent columns. The linear least squares problem

$$\min_{x \in \mathbb{R}^k} \|b - Ax\|_2$$

has a unique solution x_{min} for any $b \in \mathbb{R}^n$. The mapping $b \mapsto x_{min}$ defines a linear transformation, $A^\dagger$, from $\mathbb{R}^n$ to $\mathbb{R}^k$ called the pseudoinverse.

The pseudoinverse for rectangular full column-rank matrices behaves much as the inverse for nonsingular matrices. To see this answer the following questions and show the following identities are true :

3.9.a. Use the normal equations to write $A^\dagger$ in terms of A.

3.9.b. If $A \in \mathbb{R}^{n\times n}$ what is $A^\dagger$?

3.9.c. $AA^\dagger A = A$

3.9.d. $A^\dagger AA^\dagger = A^\dagger$

3.9.e. $A^\dagger A = (A^\dagger A)^T$

3.9.f. $AA^\dagger = (AA^\dagger)^T$

3.9.g. If $A \in \mathbb{R}^{n\times k}$ has orthonormal columns then $A^\dagger = A^T$. Why is this important for consistency with simpler forms of least squares problems that we have discussed?

Problem 3.10

Any subspace $\mathcal{S}$ of $\mathbb{R}^n$ of dimension $k \leq n$ must have at least one orthogonal matrix $Q \in \mathbb{R}^{n\times k}$ with orthonormal columns such that $\mathcal{R}(Q) = \mathcal{S}$, The matrix $P = QQ^T$ is a projector onto $\mathcal{S}$, i.e., Px is the unique component of x contained in $\mathcal{S}$.

3.10.a. P is clearly symmetric, show that it is idempotent, i.e., $P^2 = P$.

3.10.b. Show that $\mathcal{R}(P) = \mathcal{S}$.

3.10.c. Show that if M is an idempotent symmetric matrix then it is a projector onto $\mathcal{R}(M)$.

3.10.d. Choose any subspace of $\mathbb{R}^3$ that has dimension 2 and construct two orthonormal bases, Q_1 and Q_2. Verify that $P = Q_1Q_1^T = Q_2Q_2^T$.

Problem 3.11

(Problem 7.1.5 Golub and Van Loan (3rd Ed.), p. 318, Problem 7.1.5 Golub and Van Loan (4th Ed.), p. 355)

Use the Schur decomposition of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$, to show that for every $\epsilon > 0$ there exists a diagonalizable matrix B such that $\|A - B\|_2 < \epsilon$. **(Hint: consider a simple condition on the eigenvalues of B that guarantee it is diagonalizable.)**

Problem 3.12

Let $x \in \mathbb{R}^n$ be a known vector with components $\xi_i = e_i^T x,\ 1 \leq i \leq n$ and consider the computation of

$$\nu = \xi_1 - \|x\|_2$$

where $\|x\|_2^2 = \sum_{i=1}^{n} \xi_i^2$. (Recall this is a key computation in the production of a Householder reflector in least squares problems.) When $\xi_1 > 0$ and $\xi_1 \approx \|x\|_2$ the cancellation in the subtraction may result in a significant loss of accuracy.

Find an alternate expression for ν that does not suffer from cancellation when $\xi_1 > 0$ and $\xi_1 \approx \|x\|_2$. (**Hint:** Consider a difference of squares.)