Study Questions 4 Numerical Linear Algebra Spring 2024

Problem 4.1

For $A \in \mathbb{C}^{m \times n}$, prove that

$$
\begin{gathered}
rank(A) = r \\
\mathcal{N}(A) = span[v_{r+1}, \ldots, v_n] \\
\mathcal{R}(A) = span[u_1, \ldots, u_r] \\
A = \sum_{i=1}^{n} u_i \sigma_i v_i^H \\
\|A\|_2 = \sigma_1 \\
\|A\|_F^2 = \sigma_1 + \cdots + \sigma_r \\
\min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_n
\end{gathered}
$$

where $u_i = Ue_i$, $v_i = Ve_i$, $\sigma_i = e_i^T \Sigma e_i$, and $A = U\Sigma V^H$ is the SVD of A.

Problem 4.2

(Golub and Van Loan Problem 2.5.5 (3rd Ed.) p. 74, Golub and Van Loan Problem 2.4.2. (4th Ed.) p. 80)

Let $A \in \mathbb{R}^{m \times n}$ and show that

$$
\max_{x \in \Re^n,\, y \in \Re^m} \frac{y^T A x}{\|x\|_2 \|y\|_2} = \sigma_1
$$

where σ_1 is the largest singular value of A.

Problem 4.3

Given that we know the SVD exists for any complex matrix $A \in \mathbb{C}^{m \times n}$, assume that $A \in \mathbb{R}^{m \times n}$ has rank k with $k \leq n$, i.e., A is real and it may be rank deficient, and show that the SVD of A is all real and has the form

$$
A = U \begin{pmatrix} S \\ 0 \end{pmatrix} V^T = U_k \Sigma_k V_k^T
$$

where $S \in \mathbb{R}^{n\times n}$ is diagonal with nonnegative entries,

$$U = \begin{pmatrix} U_k & U_{m-k} \end{pmatrix}, \quad U^T U = I_m$$
$$V = \begin{pmatrix} V_k & V_{n-k} \end{pmatrix}, \quad V^T V = I_n$$
$$U_k \in \mathbb{R}^{m\times k}, \text{ and } V_k \in \mathbb{R}^{n\times k}$$

Hint: Consider the relationship between the SVD and the symmetric eigenvalue decomposition.

Problem 4.4

Consider the linear least squares problem with linear constraints:

$$\min_x \|b - Ax\|_2^2 \quad \text{such that} \quad Cx = d$$

where $m \geq n \geq k$, $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times k}$, $b \in \mathbb{R}^m$, $d \in \mathbb{R}^k$, $x \in \mathbb{R}^n$, A has full column rank, and C has full row rank.

Show that the problem can be converted to an unconstrained linear least squares problem.

Problem 4.5

Suppose the matrices $A \in \mathbb{R}^{n\times k}$, $x \in \mathbb{R}^k$, $V_s \in \mathbb{R}^{k\times s}$, $n > k > s+1$, with the columns of A linearly independent, and the columns of $V_s = \begin{bmatrix} v_1 & v_2 & \dots & v_s \end{bmatrix}$ also linearly independent.

4.5.a Consider the constrained linear least squares problem,

$$\min_{x \in x_0 + \mathcal{R}(V_s)} \|b - Ax\|_2$$

where $x_0 \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$ are given. (The constraint set contains vectors of the form $x = x_0 + v$, $v \in \mathcal{R}(V_s)$). Determine a system of equations that determine the unique solution $x^* = x_0 + V_s c_s^*$ where $c_s^* \in \mathbb{R}^s$.

4.5.b Now suppose a column is added to V_s to define $V_{s+1} = \begin{bmatrix} v_1 & v_2 & \dots & v_s & v_{s+1} \end{bmatrix}$ so that the columns of V_{s+1} are also linearly independent. Determine a system of equations that determine the unique solution $\tilde{x}^* = x_0 + V_{s+1} c_{s+1}^*$, where $c_{s+1}^* \in \mathbb{R}^{s+1}$, to the modified linear least squares problem

$$\min_{x \in x_0 + \mathcal{R}(V_{s+1})} \|b - Ax\|_2.$$

4.5.c Give sufficient conditions on the columns of V_{s+1} so that the two solutions are related by

$$c_{s+1}^* = \begin{pmatrix} c_s^* \\ \gamma_{s+1}^* \end{pmatrix}$$

$$\tilde{x}^* = x_0 + V_{s+1} c_{s+1}^* = x^* + v_{s+1}\, \gamma_{s+1}^*$$

Problem 4.6

Consider the roots of unity needed for a radix-2 Cooley-Tukey version of the FFT of length $n = 2^t$

$$\hat{f} = F_n f = \frac{1}{\sqrt{n}} A_0 A_1 \dots A_{t-1} P_n f$$

where P_n is the bit reversal permutation, $A_k = I_{2^k} \otimes B_{2^{t-k}}$, $k = 0, 1, \dots, t-1$, and

$$B_r = \begin{pmatrix} I_s & \Omega_s \\ I_s & -\Omega_s \end{pmatrix}$$

$$\Omega_s = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \mu_r & 0 & \dots & 0 \\ 0 & 0 & \mu_r^2 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & \dots & 0 & \mu_r^{s-1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 1 \\ 1 & \mu_2 \end{pmatrix}, \quad \mu_r = e^{-2\pi i/r}, \ ; r = 2s$$

(4.6.a) Identify the relationships between the roots of unity needed to define each of the A_k.

(4.6.b) Describe an algorithm to compute the required roots of unity. Try to make the critical path of the computation as short as possible as a function of n since its length is the coefficient of unit roundoff in the order bound on numerical error.

Problem 4.7

Consider a Cooley-Tukey version of the FFT of length $n = 16$ that uses radix-4 rather than radix-2, i.e., at each level of the FFT, all of the DFT's of length k are split into 4 each of length $k/4$. For $n = 16$ this implies

$$\hat{f} = F_{16} f = \frac{1}{\sqrt{16}} A_0 A_1 P_{16} f$$

where P_{16} is a permutation, $A_k = I_{4^k} \otimes B_{4^{t-k}}$, $k = 0, 1, \dots, t-1$, and B_r is appropriately modified from the radix-2 version.

(4.7.a) Derive the factorization and define the A_k's and P_{16}.

(4.7.b) Discuss the scatter form of P_{16} and its inverse permutation.

(4.7.c) Give the "wiring diagram" or compuational graph for F_{16} based on a radix-4 generalization of the radix-2 butterfly node we have described in class.