Study Questions 6 Numerical Linear Algebra Spring 2024

Two Useful Facts

Recall the following fundamental theorem and definition

1. (Schur Decomposition) For any $A \in \mathbb{C}^{n\times n}$ there exists $Q \in \mathbb{C}^{n\times n}$ and $R \in \mathbb{C}^{n\times n}$ where R is an upper triangular matrix and Q is a unitary matrix, i.e., $Q^H Q = QQ^H = I$, such that $A = QRQ^H$.

2. $A \in \mathbb{C}^{n\times n}$ is a normal matrix if and only if $AA^H = A^H A$ where H indicates Hermitian transpose of a matrix which is the transpose A^T if a $A \in \mathbb{R}^{n\times n}$.

Problem 6.1

(Modified Problem 7.1.1 Golub and Van Loan (3rd Ed.) p. 318, Problem 7.1.1(a) Golub and Van Loan (4th Ed.) p. 355)

6.1.a. Suppose $U \in \mathbb{C}^{n\times n}$ is block upper triangular of the form

$$U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}, \ \ U_{11} \in \mathbb{C}^{k\times k}, \ \ U_{22} \in \mathbb{C}^{n-k\times n-k}.$$

Show that if U is normal then U is a block diagonal matrix, i.e., $U_{12} = 0$, and the diagonal blocks U_{11} and U_{22} are both normal matrices.

6.1.b. Show that if $U \in \mathbb{C}^{n\times n}$ is upper triangular and normal then it is a diagonal matrix.

6.1.c. Show that if $L \in \mathbb{C}^{n\times n}$ is lower triangular and normal then it is a diagonal matrix. (Do this without repeating the entire proof above.)

Problem 6.2

(Problem 7.1.1(c) Golub and Van Loan (4th Ed.) p. 355)

Prove that a matrix $A \in \mathbb{C}^{n\times n}$ is normal if and only if there exists a unitary matrix U such that $U^H AU$ is a diagonal matrix.

Problem 6.3

Let $A \in \mathbb{C}^{n\times n}$. Define

$$H_A = \frac{1}{2}(A + A^H) \quad \text{and} \quad S_A = \frac{1}{2}(A - A^H)$$

be the Hermitian part and skew Hermitian parts of A respectively. By definition, $A = H_A + S_A$.

Show that A is a normal matrix if and only if the matrix product of H_A and S_A commutes, i.e., $H_A S_A = S_A H_A$.

Problem 6.4

Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be normal matrices that are **not diagonal matrices**. Give a sufficient condition for the matrix product of A and B to commute, i.e., $AB = BA$.

Problem 6.5

6.5.a. Show that if $A \in \mathbb{C}^{n\times n}$ is an Hermitian matrix then it is a normal matrix with real eigenvalues.

6.5.b. Show that if $A \in \mathbb{C}^{n\times n}$ is a skew Hermitian matrix, i.e., $-A = A^H$, then it is a normal matrix with imaginary eigenvalues.

Problem 6.6

(Golub and Van Loan Problem 8.3.8. (3rd Ed.) p. 424, Golub and Van Loan Problem 8.3.7. (4th Ed.) p. 465)

Suppose $A = S + \sigma uu^T$ where S is skew symmetric ($-S = S^T$), $u \in \mathbb{R}^n$ has unit 2-norm, and $\sigma \in \mathbb{R}$. Show how to compute an orthogonal Q such that $Q^T AQ$ is tridiagonal and $Q^T u = e_1$.

Problem 6.7

Let $T \in \mathbb{R}^{n\times n}$ be a symmetric tridiagonal matrix, i.e., $e_i^T T e_j = e_j^T T e_i$ and $e_i^T T e_j = 0$ if $j < i-1$ or $j > i+1$. Consider $T = QR$ where $R \in \mathbb{R}^{n\times n}$ is an upper triangular matrix and $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix.

Recall, the nonzero structure of R was derived in class and shown to be $e_i^T R e_j = 0$ if $j < i$ (upper triangular assumption) or if $j > i+2$, i.e, nonzeros are restricted to the main diagonal and the first two superdiagonals.

(6.7.a) Show that Q has nonzero structure such that $e_i^T Q e_j = 0$ if $j < i - 1$, i.e., Q is upper Hessenberg.

(6.7.b) Show that $T_+ = RQ$ is a symmetric triagonal matrix.

(6.7.c) Prove the Lemma in the class notes that states that choosing the shift $\mu = \lambda$, where λ is an eigenvalue of T, results in a reduced T_+ with known eigenvector and eigenvalue.

Problem 6.8

(Golub and Van Loan Problem 8.3.1. (3rd Ed.) p. 423, Golub and Van Loan Problem 8.3.1. (4th Ed.) p. 465)

Suppose λ is an eigenvalue of a symmetric tridiagonal matrix T. Show that if λ has algebraic mulitplicity k, then at least $k - 1$ of T's subdiagonal elements are zero.