PERGAMON

Applied Mathematics Letters 12 (1999) 93–102

Applied Mathematics Letters

A Method for Generating Rational Interpolant Reduced Order Models of Two-Parameter Linear Systems

D. S. WEILE AND E. MICHIELSSEN
Department of Electrical and Computer Engineering
Center for Computational Electromagnetics
University of Illinois at Urbana-Champaign
1406 W. Green Street, Urbana, IL 61801, U.S.A.
dsw@decwa.ece.uiuc.edu

E. GRIMME
Intel Corporation
Santa Clara, CA 95051, U.S.A.

K. GALLIVAN
Computational Science and Engineering Program
and
Computer Science Department, Florida State University
214 Love Building, Tallahassee, FL 32306, U.S.A.

(Received and accepted May 1998)

Communicated by P. Van Dooren

Abstract—A model order reduction technique for systems depending on two parameters is developed. Given a large system model, the method generates the descriptor matrices of a system model of lower order that is a rational interpolant of the transfer function of the large system—the transfer functions have identical values and derivatives for a finite set of parameter values. The new technique is a generalization of recently developed algorithms for one-parameter systems that are based on projections onto Krylov subspaces defined by the descriptor matrices.

Keywords—Generalized Krylov subspaces, Model order reduction, Rational interpolants, Padé approximation, Two-dimensional linear systems.

1. INTRODUCTION

This study presents a moment matching, model order reduction method for two-parameter problems of the form

$$\begin{aligned}(s_1\mathbf{E}_1 + s_2\mathbf{E}_2 - \mathbf{A})\,\mathbf{x}(s_1,s_2) &= \mathbf{b}u(s_1,s_2),\\ y(s_1,s_2) &= \mathbf{c}^*\mathbf{x}(s_1,s_2),\end{aligned} \tag{1}$$

This work was supported in part by a grant from AFOSR via the MURIF program under contract F49620-96-1-0025, the National Science Foundation under Grants ECS-9502138 and CCR-9796315, a grant from the IBM corporation, and an NSF graduate fellowship.

0893-9659/99/$ - see front matter. Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(99)00063-4

where $\mathbf{A}$, $\mathbf{E}_1$, and $\mathbf{E}_2$ are $n \times n$ system descriptor matrices, $\mathbf{b}$ and $\mathbf{c}$ are input and output coupling n-vectors, u is an input, y is an output, the asterisk denotes complex conjugation, and s_1 and s_2 parameters upon which the system response depends. The theory presented here results in a characterization of a reduced order model of dimension m of the form

$$\begin{aligned}\left(s_1\hat{\mathbf{E}}_1 + s_2\hat{\mathbf{E}}_2 - \mathbf{A}\right)\mathbf{x}(s_1,s_2) &= \hat{\mathbf{b}}u(s_1,s_2),\\ \hat{y}(s_1,s_2) &= \hat{\mathbf{c}}^*\mathbf{x}(s_1,s_2),\end{aligned} \tag{2}$$

where $\hat{y}(s_1,s_2)$ matches $y(s_1,s_2)$ and its derivatives at several points in the (s_1,s_2) plane for a unit input.

Recently, significant progress has been made in model reduction of systems linearly dependent on a single parameter, see [1,2] and their references. While systems of one parameter are remarkably useful in practice as they naturally occur as Laplace transforms of linear time-invariant systems, there are important problems which do not fit this form. Two directions of generalization are required. The first direction involves relaxing the restriction that the elements of the matrix defining the systems to be reduced are linear functions of the parameter of interest. Indeed, for many important problems, the system matrices are nonlinearly or even transcendentally dependent on their parameter. The second direction entails relaxing the restriction to a single parameter.

A general $p \times p$ linear system nonlinearly dependent on two parameters takes the form

$$\begin{aligned}\mathbf{G}(s_1,s_2)\mathbf{x}_{00}(s_1,s_2) &= \tilde{\mathbf{b}}u(s_1,s_2),\\ y(s_1,s_2) &= \tilde{\mathbf{c}}^*\mathbf{x}_{00}(s_1,s_2).\end{aligned} \tag{3}$$

To accomplish model reduction on this system, a method which incorporates both directions of generalization is required. To cope with the nonlinearity, the matrix $\mathbf{G}$ may be approximated at various points in the (s_1,s_2) plane by a truncated Taylor series [3]

$$\mathbf{G}(s_1,s_2) \approx \sum_{i=0}^{I}\sum_{j=0}^{i}\mathbf{G}_{ij}s_1^j s_2^{i-j}. \tag{4}$$

Defining $\mathbf{x}_{ij} = s_1^{i-j}s_2^j\mathbf{x}_{00}$ for $0 \le j \le i \le I-1$, and substituting (4) into (3) results in a system in the form of (1) where $n = p(I^2 - I + 2)/2$,

$$\mathbf{b}^* = [\mathbf{0} \ \ \dots \ \ \mathbf{0} \ \ \tilde{\mathbf{b}}^*], \qquad \mathbf{c}^* = [\tilde{\mathbf{c}}^* \ \ \mathbf{0} \ \ \dots \ \ \mathbf{0}], \qquad \mathbf{x}^* = [\tilde{\mathbf{x}}_1^* \ \ \tilde{\mathbf{x}}_2^* \ \ \dots \ \ \tilde{\mathbf{x}}_{j-1}^*],$$

and

$$\mathbf{A} = \begin{bmatrix} & \mathbf{I}_2 & & & \\ & & \mathbf{I}_3 & & \\ & & & \ddots & \\ & & & & \mathbf{I}_I \\ -\tilde{\mathbf{G}}_0 & -\tilde{\mathbf{G}}_1 & -\tilde{\mathbf{G}}_2 & \cdots & -\tilde{\mathbf{G}}_{I-1}\end{bmatrix}, \qquad \mathbf{E}_k = \begin{bmatrix}\mathbf{E}_{k1} & & & \\ & \ddots & & \\ & & \mathbf{E}_{kI-1} & \\ & & & \mathbf{H}_k\end{bmatrix},$$

for $k = 1,2$. In these expressions, $\mathbf{I}_i$ is the $lp \times lp$ identity matrix,

$$\tilde{\mathbf{G}}_l = [\mathbf{G}_{l0} \ \ \mathbf{G}_{l1} \ \ \dots \ \ \mathbf{G}_{lI-1}], \qquad \tilde{\mathbf{x}}_l^* = [\mathbf{x}_{l0}^* \ \ \dots \ \ \mathbf{x}_{ll}^*],$$

$$\mathbf{H}_1 = \left[\mathbf{G}_{I0} \ \ \frac{1}{2}\mathbf{G}_{I1} \ \ \dots \ \ \frac{1}{2}\mathbf{G}_{II-1}\right], \mathbf{H}_2 = \left[\frac{1}{2}\mathbf{G}_{I1} \ \ \dots \ \ \frac{1}{2}\mathbf{G}_{II-1} \ \ \mathbf{G}_{II}\right],$$

and

$$\mathbf{E}_{1n} = \begin{bmatrix}\mathbf{I}_1 & & \\ & \frac{1}{2}\mathbf{I}_{n-1} & \\ & & \mathbf{0}\end{bmatrix}, \qquad \mathbf{E}_{2n} = \begin{bmatrix}\mathbf{0} & & \\ & \frac{1}{2}\mathbf{I}_{n-1} & \\ & & \mathbf{I}_1\end{bmatrix}.$$

After the linearization, the problem of efficiently producing a reduced order model such as (2) from system (1) remains. Of course, it is possible to set up equations that define the parameters of the reduced-order system by explicitly matching the moments of an expansion of (1). Such an approach is taken for problems with one parameter in the method of Asymptotic Waveform Evaluation (AWE) [4]. This approach leads to severe numerical difficulties which render the matching of higher-order moments unreliable and has been replaced for large dynamical systems by the Rational Krylov-based projection methods discussed in [2].

Therefore, in this study, the notion of a Krylov space is generalized to introduce a novel, two matrix generalized Krylov space upon which the desired two-parameter generalization of the Rational Krylov family of model reduction methods is based. The generalized technique will also be shown to reduce to the rational interpolation of the Rational Krylov family if either $\mathbf{E}_2 = \mathbf{0}$ or $\mathbf{E}_2 = \mathbf{E}_1$.

It should be noted, however, that system model (1) is not the most general two-parameter form possible. For applications such as image processing, extensive work has been done on multidimensional systems. In particular, state space forms for two-parameter systems such as the Roesser Model and the General Singular Model have been proposed and analyzed [5,6]. The parameters are often used to represent two continuous or two discrete variables. For some applications, a mixed strategy has been investigated. In a mixed model, one parameter represents continuous variable while the other represents a discrete variable [7]. The literature contains work on a variety of aspects of 2-D systems including: acceptable input sequences [8], local controllability and reachability conditions [9], and stabilization of singular 2-D systems via feedback control [7]. The transfer function used in that body of work includes a product of the two parameters that does not appear in the form used in this paper.

To the best of our knowledge, however, no work on a vector space characterization of projection-based model reduction has been attempted in the 2-D systems literature. The slightly simpler transfer function used in this paper is sufficient for several significant applications in computational electromagnetics. Extensions of the results in this paper applicable to the general 2-D transfer function will be considered in a future paper.

This paper will proceed as follows. Section 2 highlights the characteristics of Krylov-based projection methods for one-parameter linear systems. In Section 3, the basic matrices and functions used throughout the paper are defined. The relevant properties of the matrices required to generate the appropriate moments of two-parameter linear systems are derived in Section 4. Section 5 introduces the generalized Krylov space and demonstrates how it can be used to define a reduced-order model satisfying the moment-matching constraints. Finally, Section 6 briefly discusses some of the applications of this technique, and directions of development currently under investigation.

2. REVIEW OF KRYLOV-BASED PROJECTION METHODS FOR ONE-PARAMETER SYSTEMS

Analogous to the reduction of system (1) to system (2), the goal of single-parameter model order reduction methods is reducing the n dimensional system

$$\begin{aligned}(s\mathbf{E} - \mathbf{A})\,\mathbf{x} &= \mathbf{b}u,\\ y &= \mathbf{c}^*\mathbf{x} + du\end{aligned} \tag{5}$$

to an $m \ll n$ dimensional system

$$\begin{aligned}\left(s\hat{\mathbf{E}} - \hat{\mathbf{A}}\right)\hat{\mathbf{x}} &= \hat{\mathbf{b}}u,\\ y &= \hat{\mathbf{c}}^*\hat{\mathbf{x}} + du,\end{aligned} \tag{6}$$

which preserves some qualities of the original system. Projection methods for model order reduction involve characterizing rectangular matrices $\mathbf{V}$ and $\mathbf{Z}$ such that the descriptor matrices of the reduced order model (6) can be computed from the matrices in (5) as $\hat{\mathbf{A}} = \mathbf{Z}^*\mathbf{AV}$, $\hat{\mathbf{E}} = \mathbf{Z}^*\mathbf{EV}$, $\hat{\mathbf{b}} = \mathbf{Z}^*\mathbf{b}$, and $\hat{\mathbf{c}} = \mathbf{V}^*\mathbf{c}$ [2]. Most often, $\mathbf{V}$ and $\mathbf{Z}$ are chosen so that the resulting reduced-order model (6) is a partial realization, Padé approximation, or rational interpolant of the transfer function of the original system (5)

$$h(s) = \mathbf{c}^*(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}.$$

These approximants are defined by the location and number of points in the frequency domain at which $h(s)$ and its moments of specified orders match the transfer function $\hat{h}(s)$ of the reduced-order system. In [2], the projections that achieve all three of these approximations are characterized, and it is shown that in all cases, colsp($\mathbf{V}$) and colsp($\mathbf{Z}$) must contain unions of certain Krylov spaces defined in terms of the systems descriptor matrices and vectors. A Krylov space is defined by a matrix $\mathbf{G}$ and a vector $\mathbf{g}$ and is of the form

$$\mathcal{K}_j(\mathbf{G}, \mathbf{g}) = \operatorname{colsp}\left\{\bigcup_{k=0}^{j-1} \mathbf{G}^k\mathbf{g}\right\}.$$

The similarity between (1) and typical one-parameter linear systems of the form (5) indicates that a generalization of the Krylov space concept might be possible, resulting in characterizations of the spaces associated with projection-based techniques for systems with two parameters. Just as in the one-parameter case, the availability of such spaces naturally leads to algorithms for model reduction which avoid the numerical problems of explicitly matching moments as in AWE.

3. BASIC DEFINITIONS

Consider a linear system described by (1). Solving (1) for y gives

$$y = h(s_1, s_2)u, \tag{7}$$

where the two-parameter transfer function $h(s_1, s_2)$ may be written as

$$h(s_1, s_2) = \left[\mathbf{c}^*\left(s_1\mathbf{E}_1 + s_2\mathbf{E}_2 - \mathbf{A}\right)^{-1}\mathbf{b}\right]. \tag{8}$$

Via a shift of coordinates $(s_1, s_2) \to (s_1 - \sigma_1, s_2 - \sigma_2)$ and a Taylor expansion, (8) may be rewritten as

$$h(s_1, s_2) = -\mathbf{c}^* \sum_{j=0}^{\infty} \left[s_1\mathbf{P}^{-1}\mathbf{E}_1 + s_2\mathbf{P}^{-1}\mathbf{E}_2\right]^j \mathbf{P}^{-1}\mathbf{b}, \tag{9}$$

where

$$\mathbf{P} = \mathbf{A} - \sigma_1\mathbf{E}_1 - \sigma_2\mathbf{E}_2, \tag{10}$$

for some complex σ_1 and σ_2. Expanding each term in (9), and denoting the matrix multiplying the scalar $s_1^{j-k}s_2^k$ as $\mathbf{F}_k^j(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)$, (9) may be rewritten as

$$h(s_1, s_2) = -\mathbf{c}^* \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left[\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right) s_1^{j-k}s_2^k\right]\mathbf{P}^{-1}\mathbf{b}. \tag{11}$$

Because the function $h(s_1, s_2)$ is the two-parameter generalization of the transfer function in the one-parameter case, the scalar values

$$-\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b}$$

may be thought of as transfer function moments. The functions $\mathbf{F}_k^j$ are, therefore, referred to as moment generating functions due to their intimate relationship with series (11). The goal of projection-based model reduction is to produce $n \times m$ matrices $\mathbf{Z}$ and $\mathbf{V}$ that define projections such that reduced order model (2) can be written in terms of system (1) with

$$\begin{gathered} \hat{\mathbf{A}} = \mathbf{Z}^*\mathbf{AV}, \qquad \hat{\mathbf{E}}_1 = \mathbf{Z}^*\mathbf{E}_1\mathbf{V}, \qquad \hat{\mathbf{E}}_2 = \mathbf{Z}^*\mathbf{E}_2\mathbf{V}, \\ \hat{\mathbf{b}} = \mathbf{Z}^*\mathbf{b}, \qquad \text{and} \qquad \hat{\mathbf{c}} = \mathbf{V}^*\mathbf{c}, \end{gathered} \tag{12}$$

such that the transfer function of (2) matches values and selected moments of the transfer function of (1).

4. PROPERTIES OF THE MOMENT GENERATING FUNCTIONS

Given the definition of the moment generating functions $\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)$, a method is required for calculating them recursively for arbitrary square matrices $\mathbf{G}_1$ and $\mathbf{G}_2$. To facilitate this, define

$$\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2) = 0, \qquad \forall k \notin \{0, 1, \ldots, j\}$$

as these $\mathbf{F}_k^j$ do not appear in the expansion of $(s_1\mathbf{G}_1 + s_2\mathbf{G}_2)^j$. Then, the $\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)$ can be generated with the following theorem.

THEOREM 1. *Recursive generation of* $\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)$.

$$\begin{aligned} \mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2) &= \mathbf{G}_2\mathbf{F}_{k-1}^{j-1}(\mathbf{G}_1, \mathbf{G}_2) + \mathbf{G}_1\mathbf{F}_k^{j-1}(\mathbf{G}_1, \mathbf{G}_2) \\ &= \mathbf{F}_{k-1}^{j-1}(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{G}_2 + \mathbf{F}_k^{j-1}(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{G}_1, \qquad j = 1, 2, \ldots. \end{aligned} \tag{13}$$

PROOF. By induction. Note that by definition,

$$\begin{aligned} \mathbf{F}_0^0(\mathbf{G}_1, \mathbf{G}_2) &= \mathbf{I}, \\ \mathbf{F}_0^1(\mathbf{G}_1, \mathbf{G}_2) &= \mathbf{G}_1, \\ \mathbf{F}_1^1(\mathbf{G}_1, \mathbf{G}_2) &= \mathbf{G}_2. \end{aligned}$$

Thus, the theorem holds for $j = 1$ for all k. Now, assume (13) holds for all $j < J$. Note that

$$\begin{aligned} (s_1\mathbf{G}_1 + s_2\mathbf{G}_2)^J &= (s_1\mathbf{G}_1 + s_2\mathbf{G}_2)\sum_{k=0}^{J-1} s_1^{J-1-k}s_2^k\mathbf{F}_k^{J-1}(\mathbf{G}_1, \mathbf{G}_2) \\ &= \sum_{k=0}^{J} s_1^{J-k}s_2^k\mathbf{F}_k^J(\mathbf{G}_1, \mathbf{G}_2) \end{aligned} \tag{14}$$

or, alternatively,

$$\begin{aligned} (s_1\mathbf{G}_1 + s_2\mathbf{G}_2)^J &= \sum_{k=0}^{J-1} s_1^{J-1-k}s_2^k\mathbf{F}_k^{J-1}(\mathbf{G}_1, \mathbf{G}_2)\,(s_1\mathbf{G}_1 + s_2\mathbf{G}_2) \\ &= \sum_{k=0}^{J} s_1^{J-k}s_2^k\mathbf{F}_k^J(\mathbf{G}_1, \mathbf{G}_2) \end{aligned} \tag{15}$$

by definition. Matching coefficients in $s_1^{J-k}s_2^k$ in (14) gives the first equality; doing the same in (15) gives the second. ∎

The equivalence

$$\mathbf{G}_2\mathbf{F}_{k-1}^{j-1}(\mathbf{G}_1, \mathbf{G}_2) + \mathbf{G}_1\mathbf{F}_k^{j-1}(\mathbf{G}_1, \mathbf{G}_2) = \mathbf{F}_{k-1}^{j-1}(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{G}_2 + \mathbf{F}_k^{j-1}(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{G}_1 \tag{16}$$

established by Theorem 1 is referred to as pseudo-commutativity, because the positioning of the matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ with respect to the moment generating functions is reminiscent of commutativity between arbitrary powers of a matrix in moment generation for one-parameter systems. This should not be misconstrued as the generally incorrect statement that $\mathbf{G}_1$ and $\mathbf{G}_2$ commute with the moment generation matrices. Theorem 2 shows that there is a pseudo-associativity as well.

THEOREM 2. *Pseudo-Associativity.*

$$\mathbf{F}_k^j(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2)\mathbf{B} = \mathbf{B}\mathbf{F}_k^j(\mathbf{G}_1\mathbf{B}, \mathbf{G}_2\mathbf{B}), \qquad j = 0, 1, \dots. \tag{17}$$

PROOF. By induction. If $j = 0$, the result is trivially true for all k. Now assume it is true for $j < J$ and all k and note that

$$\begin{aligned}\mathbf{F}_k^J(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2)\mathbf{B} &= \left\{\mathbf{B}\mathbf{G}_2\mathbf{F}_{k-1}^{J-1}(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2) + \mathbf{B}\mathbf{G}_1\mathbf{F}_k^{J-1}(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2)\right\}\mathbf{B} \\ &= \mathbf{B}\left\{\mathbf{G}_2\mathbf{F}_{k-1}^{J-1}(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2)\mathbf{B} + \mathbf{G}_1\mathbf{F}_k^{J-1}(\mathbf{B}\mathbf{G}_1, \mathbf{B}\mathbf{G}_2)\mathbf{B}\right\} \\ &= \mathbf{B}\left\{\mathbf{G}_2\mathbf{B}\mathbf{F}_{k-1}^{J-1}(\mathbf{G}_1\mathbf{B}, \mathbf{G}_2\mathbf{B}) + \mathbf{G}_1\mathbf{B}\mathbf{F}_k^{J-1}(\mathbf{G}_1\mathbf{B}, \mathbf{G}_2\mathbf{B})\right\} \\ &= \mathbf{B}\mathbf{F}_k^j(\mathbf{G}_1\mathbf{B}, \mathbf{G}_2\mathbf{B}).\end{aligned}$$

The third step follows from the induction hypothesis; the last from Theorem 1. ∎

Finally, Theorem 3 shows that the moment generation matrices $\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)$ at any level j may be generated from the moment generation matrices at levels l and $j - l$ for $l < j$.

THEOREM 3. *Generalized recursive generation of* $\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)$.

$$\mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2) = \sum_{i=0}^{l}\mathbf{F}_{l-i}^l(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{k-l+i}^{j-l}(\mathbf{G}_1, \mathbf{G}_2), \qquad 0 \le l \le j. \tag{18}$$

PROOF. By induction. If $l = 0$, then expression (18) is trivially true for all k. Assume (18) holds for $l \le L$. Then notice that

$$\begin{aligned}&\sum_{i=0}^{L+1}\mathbf{F}_{L+1-i}^{L+1}(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{k-(L+1)+i}^{j-(L+1)}(\mathbf{G}_1, \mathbf{G}_2) \\ &\quad = \sum_{i=0}^{L+1}\left[\mathbf{G}_2\mathbf{F}_{L-i}^L(\mathbf{G}_1, \mathbf{G}_2) + \mathbf{G}_1\mathbf{F}_{L+1-i}^L(\mathbf{G}_1, \mathbf{G}_2)\right]\mathbf{F}_{k-(L+1)+i}^{j-(L+1)}(\mathbf{G}_1, \mathbf{G}_2) \\ &\quad = \mathbf{G}_2\sum_{i=0}^{L+1}\mathbf{F}_{L-i}^L(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{F}_{k-(L+1)+i}^{j-(L+1)}(\mathbf{G}_1, \mathbf{G}_2) \\ &\qquad + \mathbf{G}_1\sum_{i=0}^{L+1}\mathbf{F}_{L+1-i}^L(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{k-(L+1)+i}^{j-(L+1)}(\mathbf{G}_1, \mathbf{G}_2).\end{aligned} \tag{19}$$

Using the fact that $\mathbf{F}_{-1}^L(\mathbf{G}_1, \mathbf{G}_2) = \mathbf{F}_{L+1}^L(\mathbf{G}_1, \mathbf{G}_2) = \mathbf{0}$, and incorporating a shift of index in the second sum, (19) becomes

$$\begin{aligned}&\sum_{i=0}^{L+1}\mathbf{F}_{L+1-i}^{L+1}(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{k-(L+1)+i}^{j-(L+1)}(\mathbf{G}_1, \mathbf{G}_2) \\ &\quad = \mathbf{G}_2\sum_{i=0}^{L}\mathbf{F}_{L-i}^L(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{(k-1)-L+i}^{(j-1)-L}(\mathbf{G}_1, \mathbf{G}_2) + \mathbf{G}_1\sum_{i=0}^{L}\mathbf{F}_{L-i}^L(\mathbf{G}_1, \mathbf{G}_2)\mathbf{F}_{k-L+i}^{(j-1)-L}(\mathbf{G}_1, \mathbf{G}_2) \\ &\quad = \mathbf{G}_2\mathbf{F}_{k-1}^{j-1}(\mathbf{G}_1, \mathbf{G}_2) + \mathbf{G}_1\mathbf{F}_k^{j-1}(\mathbf{G}_1, \mathbf{G}_2) = \mathbf{F}_k^j(\mathbf{G}_1, \mathbf{G}_2)\end{aligned}$$

by the induction hypothesis and Theorem 1.

5. GENERALIZED KRYLOV SPACES AND MOMENT MATCHING

Now that methods for calculating the moment matching functions have been established, generalized, two matrix Krylov spaces can be defined that contain information relevant to the moments of system (1). A two matrix, generalized Krylov subspace $W_j(\mathbf{G}_1, \mathbf{G}_2, \mathbf{g})$ may be defined as

$$W_j(\mathbf{G}_1, \mathbf{G}_2, \mathbf{g}) = \operatorname{colsp}\left\{ \bigcup_{m=0}^{j} \left(\bigcup_{k=0}^{m} \mathbf{F}_k^m(\mathbf{G}_1, \mathbf{G}_2)\,\mathbf{g} \right) \right\}.$$

The following lemmas provide the basis for a model reduction theorem.

LEMMA 1. *If* $\mathbf{W}^*\mathbf{V} = \mathbf{I}$ *and* $\mathbf{v} \in \operatorname{colsp}\{\mathbf{V}\}$ *then* $\mathbf{V}\mathbf{W}^*\mathbf{v} = \mathbf{v}$ *(see [2]).*

PROOF. $\mathbf{v} \in \operatorname{colsp}\{\mathbf{V}\} \to \mathbf{v} = \mathbf{V}\mathbf{g}$. Then $\mathbf{V}\mathbf{W}^*\mathbf{v} = \mathbf{V}\mathbf{W}^*\mathbf{V}\mathbf{g} = \mathbf{V}\mathbf{g} = \mathbf{v}$. ∎

For the lemmas that follow, define $\hat{\mathbf{P}} = \mathbf{Z}^*\mathbf{P}\mathbf{V}$ in concert with the definitions in (12). Furthermore, let $\mathbf{G}^{-*} = (\mathbf{G}^*)^{-1}$ for any nonsingular matrix $\mathbf{G}$.

LEMMA 2. *If* $W_J(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2, \mathbf{P}^{-1}\mathbf{b}) \subseteq \operatorname{colsp}\{\mathbf{V}\}$ *and* $\mathbf{Z}$ *is any* $n \times m$ *matrix, then*

$$\mathbf{F}_k^j(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\,\mathbf{P}^{-1}\mathbf{b} = \mathbf{V}\mathbf{F}_k^j\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}, \qquad 0 \le k \le j \le J. \tag{21}$$

PROOF. By induction. First, define $\mathbf{W}^* = (\mathbf{Z}^*\mathbf{P}\mathbf{V})^{-1}\mathbf{Z}^*\mathbf{P}$ so that $\mathbf{W}^*\mathbf{V} = \mathbf{I}$. Now examine the case $j = k = 0$. Starting with the right-hand side of (21)

$$\begin{aligned}\mathbf{V}\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}} &= \mathbf{V}(\mathbf{Z}^*\mathbf{P}\mathbf{V})^{-1}\mathbf{Z}^*\mathbf{b}\\ &= \mathbf{V}(\mathbf{Z}^*\mathbf{P}\mathbf{V})^{-1}\mathbf{Z}^*\mathbf{P}\mathbf{P}^{-1}\mathbf{b}\\ &= \mathbf{V}\mathbf{W}^*\mathbf{P}^{-1}\mathbf{b}\\ &= \mathbf{P}^{-1}\mathbf{b}\end{aligned}$$

by Lemma 4. Assume now that (21) holds for $0 \le k \le j-1 \le J-1$. Then

$$\begin{aligned}&\mathbf{V}\mathbf{F}_k^j\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}\\ &\quad= \mathbf{V}\left[\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\mathbf{F}_{k-1}^{j-1}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\right.\\ &\qquad\left.+\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1\mathbf{F}_k^{j-1}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\right]\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}\\ &\quad= \mathbf{V}\left\{\hat{\mathbf{P}}^{-1}\mathbf{Z}^*\mathbf{E}_2\left[\mathbf{V}\mathbf{F}_{k-1}^{j-1}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}\right]\right.\\ &\qquad\left.+\hat{\mathbf{P}}^{-1}\mathbf{Z}^*\mathbf{E}_1\left[\mathbf{V}\mathbf{F}_k^{j-1}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}\right]\right\}\\ &\quad= \mathbf{V}\left\{\hat{\mathbf{P}}^{-1}\mathbf{Z}^*\mathbf{E}_2\left[\mathbf{F}_{k-1}^{j-1}(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\,\mathbf{P}^{-1}\mathbf{b}\right]\right.\\ &\qquad\left.+\hat{\mathbf{P}}^{-1}\mathbf{Z}^*\mathbf{E}_1\left[\mathbf{F}_k^{j-1}(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\,\mathbf{P}^{-1}\mathbf{b}\right]\right\}\end{aligned} \tag{22}$$

by the inductive hypothesis. By removing common factors, expression (22) may be then written as

$$\begin{aligned}&\mathbf{V}\mathbf{F}_k^j\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}\\ &\quad= \mathbf{V}(\mathbf{Z}^*\mathbf{P}^{-1}\mathbf{V})\,\mathbf{Z}^*\mathbf{P}\left[\mathbf{P}^{-1}\mathbf{E}_2\mathbf{F}_{k-1}^{j-1}(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\right.\\ &\qquad\left.+\mathbf{P}^{-1}\mathbf{E}_1\mathbf{F}_k^{j-1}(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\right]\mathbf{P}^{-1}\mathbf{b}\\ &\quad= \mathbf{V}\mathbf{W}^*\mathbf{F}_k^j(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\,\mathbf{P}^{-1}\mathbf{b}\\ &\quad= \mathbf{F}_k^j(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2)\,\mathbf{P}^{-1}\mathbf{b}\end{aligned}$$

by Theorem 1 and Lemma 1. ∎

LEMMA 3. *If* $W_J(\mathbf{P}^{-*}\mathbf{E}_1, \mathbf{P}^{-*}\mathbf{E}_2, \mathbf{P}^{-*}\mathbf{c}) \subseteq \mathrm{colsp}\{\mathbf{Z}\}$, *and* $\mathbf{V}$ *is any* $n \times m$ *matrix, then*

$$\mathbf{c}^*\mathbf{P}^{-1}\mathbf{F}_k^j\left(\mathbf{E}_1\mathbf{P}^{-1}, \mathbf{E}_2\mathbf{P}^{-1}\right) = \hat{\mathbf{c}}^*\hat{\mathbf{P}}^{-1}\mathbf{F}_k^j\left(\hat{\mathbf{E}}_1\hat{\mathbf{P}}^{-1}, \hat{\mathbf{E}}_2\hat{\mathbf{P}}^{-1}\right)\mathbf{Z}^*, \qquad 0 \le k \le j \le J.$$

The proof of Lemma 3 is dual to that of Lemma 2. With these lemmas demonstrated, a theorem for generating a reduced order model of (1) in the form of (11) and (12) is now proven.

THEOREM 4. MODEL REDUCTION. *If* $W_{J_b}(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2, \mathbf{P}^{-1}\mathbf{b}) \subseteq \mathrm{colsp}\{\mathbf{V}\}$ *and* $W_{J_c}(\mathbf{P}^{-*}\mathbf{E}_1, \mathbf{P}^{-*}\mathbf{E}_2, \mathbf{P}^{-*}\mathbf{c}) \subseteq \mathrm{colsp}\{\mathbf{Z}\}$ *then*

$$\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} = \hat{\mathbf{c}}^*\mathbf{F}_k^j\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}, \tag{24}$$

for $0 \le k \le j \le J_b + J_c + 1$.

PROOF. The $j = 0$ case follows immediately from the lemmas, so it is not proven here. Thus, choose $j_b \le J_b$ and $j_c \le J_c$ such that $j_b + j_c + 1 = j$. Now

$$\begin{aligned}
&\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} \\
&\quad = \mathbf{c}^*\sum_{i=0}^{j_c+1}\mathbf{F}_{j_c+1-i}^{j_c+1}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{F}_{k-j_c-1+i}^{j_b}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b}
\end{aligned} \tag{25}$$

by Theorem 3. Applying Theorem 1, (25) may be written as

$$\begin{aligned}
&\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} \\
&= \sum_{i=0}^{j_c+1}\mathbf{c}^*\Big[\mathbf{F}_{j_c-i}^{j_c}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{E}_2 \\
&\quad + \mathbf{F}_{j_c+1-i}^{j_c}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{E}_1\Big]\mathbf{F}_{k-j_c-1+i}^{j_b}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} \\
&= \sum_{i=0}^{j_c+1}\Big[\mathbf{c}^*\mathbf{P}^{-1}\mathbf{F}_{j_c-i}^{j_c}\left(\mathbf{E}_1\mathbf{P}^{-1}, \mathbf{E}_2\mathbf{P}^{-1}\right)\mathbf{E}_2 + \mathbf{c}^*\mathbf{P}^{-1}\mathbf{F}_{j_c+1-i}^{j_c}\left(\mathbf{E}_1\mathbf{P}^{-1}, \mathbf{E}_2\mathbf{P}^{-1}\right)\mathbf{E}_1\Big] \\
&\quad \times \mathbf{F}_{k-j_c-1+i}^{j_b}\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b}
\end{aligned} \tag{26}$$

by pseudo-associativity. The moment matching Lemmas 2 and 3 may be inserted into (26) to rewrite it in terms of reduced model parameters to yield

$$\begin{aligned}
&\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1, \mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} \\
&= \sum_{i=0}^{j_c+1}\Big[\hat{\mathbf{c}}^*\hat{\mathbf{P}}^{-1}\mathbf{F}_{j_c-i}^{j_c}\left(\hat{\mathbf{E}}_1\hat{\mathbf{P}}^{-1}, \hat{\mathbf{E}}_2\hat{\mathbf{P}}^{-1}\right)\mathbf{Z}^*\mathbf{E}_2 + \hat{\mathbf{c}}^*\hat{\mathbf{P}}^{-1}\mathbf{F}_{j_c+1-i}^{j_c}\left(\hat{\mathbf{E}}_1\hat{\mathbf{P}}^{-1}, \hat{\mathbf{E}}_2\hat{\mathbf{P}}^{-1}\right)\mathbf{Z}^*\mathbf{E}_1\Big] \\
&\quad \times \mathbf{V}\mathbf{F}_{k-j_c-1+i}^{j_b}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}} \\
&= \sum_{i=0}^{j_c+1}\hat{\mathbf{c}}^*\Big[\hat{\mathbf{P}}^{-1}\mathbf{F}_{j_c-i}^{j_c}\left(\hat{\mathbf{E}}_1\hat{\mathbf{P}}^{-1}, \hat{\mathbf{E}}_2\hat{\mathbf{P}}^{-1}\right)\hat{\mathbf{E}}_2 + \hat{\mathbf{P}}^{-1}\mathbf{F}_{j_c+1-i}^{j_c}\left(\hat{\mathbf{E}}_1\hat{\mathbf{P}}^{-1}, \hat{\mathbf{E}}_2\hat{\mathbf{P}}^{-1}\right)\hat{\mathbf{E}}_1\Big] \\
&\quad \times \mathbf{F}_{k-j_c-1+i}^{j_b}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}} \\
&= \sum_{i=0}^{j_c+1}\hat{\mathbf{c}}^*\Big[\mathbf{F}_{j_c-i}^{j_c}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2 + \mathbf{F}_{j_c+1-i}^{j_c}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1\Big] \\
&\quad \times \mathbf{F}_{k-j_c-1+i}^{j_b}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1, \hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}
\end{aligned} \tag{27}$$

by pseudo-associativity. Invoking pseudo-commutativity, (27) becomes

$$\begin{aligned} &\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1,\mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} \\ &= \sum_{i=0}^{j_c+1} \hat{\mathbf{c}}^*\mathbf{F}_{j_c+1-i}^{j_c+1}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1,\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\mathbf{F}_{k-j_c-1+i}^{j_b}\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1,\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}, \end{aligned} \tag{28}$$

and therefore, by Theorem 3, (28) becomes

$$\mathbf{c}^*\mathbf{F}_k^j\left(\mathbf{P}^{-1}\mathbf{E}_1,\mathbf{P}^{-1}\mathbf{E}_2\right)\mathbf{P}^{-1}\mathbf{b} = \hat{\mathbf{c}}^*\mathbf{F}_k^j\left(\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_1,\hat{\mathbf{P}}^{-1}\hat{\mathbf{E}}_2\right)\hat{\mathbf{P}}^{-1}\hat{\mathbf{b}}. \qquad \blacksquare$$

This theorem establishes that a reduced order model constructed with $\mathbf{V}$ and $\mathbf{Z}$ given above constitute a system with a transfer function that is a Padé approximant to the true system transfer function. To instead accomplish a rational interpolant which matches all of the moments of orders between 0 and $J_b^i + J_c^i + 1$ at ordered pairs (σ_1^i, σ_2^i) $i = 1, \dots, K$ for some number of interpolation points K, then $\mathbf{V}$ and $\mathbf{Z}$ need to be constructed so that

$$\bigcup_{i=1}^{K} W_{J_b^i}\left(\mathbf{P}_i^{-1}\mathbf{E}_1, \mathbf{P}_i^{-1}\mathbf{E}_2, \mathbf{P}_i^{-1}\mathbf{b}\right) \subseteq \operatorname{colsp}\{\mathbf{V}\}$$

and

$$\bigcup_{i=1}^{K} W_{J_c^i}\left(\mathbf{P}_i^{-*}\mathbf{E}_1, \mathbf{P}_i^{-*}\mathbf{E}_2, \mathbf{P}_i^{-*}\mathbf{c}\right) \subseteq \operatorname{colsp}\{\mathbf{Z}\}.$$

The theorem then shows that such spaces contain the needed information to generate the desired model.

6. CONCLUSIONS

A generalization for two-parameter linear systems of the vector spaces and projections used to form reduced-order moment-matched models for one-parameter linear systems contained in [2] has been presented. The spaces have been characterized via a generalization of the standard Krylov space, and the matrices $\mathbf{V}$ and $\mathbf{Z}$ that perform projections onto the appropriate spaces have been derived via a simple recursion. We note that theorems demonstrating the moment-matching properties of the reduced order model do not depend on the method of construction of the matrices. In practice, there are many algorithms that can be used to produce $\mathbf{V}$ and $\mathbf{Z}$. For instance, the Rational Krylov family for one-parameter problems includes: the multipoint Rational Arnoldi, Rational Lanczos, Dual Rational Arnoldi, and Rational Power methods. For two-parameter problems, the basic recursions that define $\mathbf{V}$ and $\mathbf{Z}$ have been used along with a simple orthogonalization strategy to produce a generalization of the Dual Rational Arnoldi Algorithm for two parameter problems. Generalizations of the other members of the family are under consideration.

The combination of linearization and model reduction relative to two parameters using the generalized Dual Rational Arnoldi algorithm has been successfully applied to significant applications in electromagnetics. Specifically, the method has been applied to the analysis frequency selective surfaces which are used for frequency and angular filtering and serve as satellite subreflectors for dual band receivers [10]. Such problems are formulated using well-known integral equation techniques, and result in systems of the form of (4) where the parameters s_1 and s_2 correspond to the frequency and incident angle of an electromagnetic wave impinging on the frequency selective surfaces. This method is also being extended to more general scattering problems. These applications of the theory and generalized Rational Krylov family methods will be discussed in forthcoming papers.

REFERENCES

1. K. Gallivan, E. Grimme and P.V. Dooren, A rational Lanczos method for model reduction, *Numerical Algorithms* **12** (1), 33–63, (1996).
2. E.J. Grimme, Krylov projection methods for model reduction, Ph.D. Thesis, University of Illinois at Urbana-Champaign, (1997).
3. I. Elfadel, J. Phillips, M. Silveira and J. White, A brief survey of Krylov-subspace based model order reduction, In *Proceedings of the North American Radio Science Meeting*, p. 147, Montreal, (1997).
4. E. Chiprout and M.S. Nakhla, *Asymptotic Waveform Evaluation and Moment Matching for Interconnect Analysis*, Kluwer Academic, Boston, (1994).
5. R.P. Roesser A discrete state-space model for linear image processing, *IEEE Transactions on Automation and Control* **AC-20** (1), 1–10, (1975).
6. T. Kaczorek, General response formula and minimum energy control for the general singular model of 2-D systems, *IEEE Transactions on Automation and Control* **35** (4), 433–436, (1990).
7. T. Kaczorek, Stabilization of singular 2-D continuous-discrete systems by state feedback controllers, *IEEE Transactions on Automation and Control* **41** (7), 1007–1009, (1996).
8. T. Kaczorek, Acceptable input sequences for singular 2-D linear systems, *IEEE Transactions on Automation and Control* **38** (9), 1391–1394, (1993).
9. T. Kaczorek, Local controllability, reachability and reconstructability of the general singular model of 2-D systems, *IEEE Transactions on Automation and Control* **37** (10), 1527–1530, (1992).
10. R. Mittra, C. H. Chan and T. Cwik, Techniques for analyzing frequency selective surfaces—A review, *Proceedings of the IEEE* **76** (12), 1593–1615, (1988).