Projection Methods for Model Reduction of Large-scale Systems

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Topics

- Brief summary of previous work in computational science and engineering
- Model reduction of a system represented by:
  - a differential equation
  - a differential equation and data
  - data, a parameterization and an optimization metric

Research Goal

To understand the interaction and tradeoffs between the

- Algorithms
- Applications
- Architectures (software and hardware, small to large scale)

to create systems that contribute to science and engineering

Activities

- develop efficient and reliable algorithms
- develop theory that yields algorithmic and implementation insights
- develop analytical and empirical modeling to
  - identify the appropriate architecture to achieve performance goals
  - adapt and tune the version of the algorithm for that architecture/application combination
- create proof-of-concept, research-grade, pre-production software
- assist in the improvement of the architectures (software and hardware)
- create software systems that facilitate all of the above
- look for similarity between applications for technology transfer

Example of multidisciplinary activity over time

Algorithm Code	Performance Evaluation	Compiler
BLAS3	kernel/cache/ parallel model	
Dense linear algebra	load/store model	window analysis
Structured problems	Perfect Club analysis	FALCON

Current: Cache models and iterative compilation

Chains of Recurrences for performance restructuring

Technique	Code
Parallel Direct Method	PY12M
Reordering,Pivoting for multiclusters	MCSPARSE, H*
Itetative/Direct hybrid (nonsymmetric)	EN-like methods PARASPAR
Sparse Least Squares	CIMGS
LTI model reduction	RK Family
Large-scale stabilization	Riccati-based TSQR Family
Parallel MG coarsening	BoomerAMG addition

Application/Architecture/Algorithm Interaction

Perfect Club – 'hand-tuned' versions (constructs still relevant in restructuring)

GFDL-IMGA Ocean Circulation model – Mediterranean basin, multicluster mapping

Electromagnetics – Frequency selective screen simulation, extended model reduction algorithms to two-parameters and additional frequency dependece

Circuit simulation – Hierarchical relaxation (waveform, nonlinear, linear), extensive implementation analysis of memory system, control system and runtime library of Cedar starting point for model reduction

Current Application/Algorithm Interaction

- Geodesic searches for image recognition (Liu, Srivastava)
- Autofocus for synthetic aperture radar (Munson)
- Efficient high-fidelity evaluation of models with stochastic parameters (Hussaini)

Collaborators

- Model Reduction: P. Van Dooren, E. Grimme, A. Van den Dorpe
- Incremental Dominant Spaces: P. Van Dooren, Y. Chahlaoui, C. Baker
- Grassmann and Projection Search: A. Srivastava, X. Liu

Goal

Efficiently approximate a large-scale dynamical system with a reduced-order model that is

- accurate in response and form.
- smaller in dimension and faster to use relative to some task.
- can be produced efficiently relative to the amount preprocessing time allowed by subsequent use

continuous-time

$$\begin{cases} \dot{x}(t) = G(x(t), u(t)) \\ y(t) = H(x(t), u(t)) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}(t) = A(t)x(t)+B(t)u(t) \\ y(t) = C(t)x(t)+D(t)u(t) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

discrete-time

$$\begin{cases} x(k+1) = G(x(k), u(k)) \\ y(k) = H(x(k), u(k)) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x(k+1) = A(k)x(k)+B(k)u(k) \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Last two cases have extensively been studied

Linear Time Invariant Systems (Implicit)

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t). \end{cases}$$

$u(t) \in \Re^m$, $y(t) \in \Re^p$, $x(t) \in \Re^N$, $N >> m, p$

Find another system driven with the same input

$$\begin{cases} \hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t), \end{cases}$$

but $\hat{y}(t) \in \Re^p$, $\hat{x}(t) \in \Re^n$.

Find low order model with small output error $\|y(t) - \hat{y}(t)\|$

Uses

- accelerate simulation of a large system, e.g., exploiting large LTI interconnect subsystem in nonlinear ODEs for circuit simulation
- accelerate simulation of a large LTI system that represents a peturbation to a solution of interest for a large nonlinear system
- preprocessing to create a smaller computationally tractably problem that is solved and whose solution is “lifted” back to the large space, e.g., stabilizing feedback

What kind of approximations?

Transfer functions

$$T(s) = C(sI_N - A)^{-1}B + D, \quad \hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D},$$

Infinity norm

$$\|T(.) - \hat{T}(.)\|_\infty \doteq \sup_\omega \|T(j\omega) - \hat{T}(j\omega)\|_2,$$

Hankel norm

$$\|T(.) - \hat{T}(.)\|_{\mathcal{H}}^2 \doteq \sigma_i^2(\mathcal{H})$$

Balanced truncation

Apply a balancing state space transformation and truncate to leading $n \times n$ block of transformed A (and conformally on other matrices) This is often used due to bound relative to the $\| \ . \ \|_\infty$ norm.

Modal Approximation

Suppose for simplicity that all the poles of $T(s)$ are different.

$$
\begin{aligned}
T(s) &= \kappa \frac{(s-\beta_1)\ldots(s-\beta_{N-1})}{(s-\alpha_1)\ldots(s-\alpha_N)} \\
&= \frac{\gamma_1}{s-\alpha_1} + \ldots + \frac{\gamma_N}{s-\alpha_N}
\end{aligned}
$$

Idea : Keep the k dominant modes of the partial fraction expansion

$$
\hat{T}(s) = \frac{\gamma_1}{s-\alpha_1} + \ldots + \frac{\gamma_k}{s-\alpha_k}.
$$

Advantages : Preserves stability, not always easy to determine which poles are dominant in an efficient manner for large systems

Rational interpolation / moment matching

Let $T(s) = C(sE - A)^{-1}B$. Consider the interpolation set

$$I = \{(\sigma_1, k_1), \ldots, (\sigma_r, k_r)\}.$$

construct a reduced order transfer function

$$\hat{T}(s) = \hat{C}(sI_k - \hat{A})^{-1}\hat{B},$$

that interpolates $R(s)$ at σ_i up to the k_i-th Taylor expansion term $\forall\ 1 \leq i \leq r,\ \forall\ 1 \leq j \leq k_i$:

$$\begin{aligned} \frac{d^{j-1}}{ds^{j-1}}\{R(s)\}\bigg|_{s=\sigma_i} &= \frac{d^{j-1}}{ds^{j-1}}\{\hat{R}(s)\}\bigg|_{s=\sigma_i} \\ \Longleftrightarrow \hat{C}(\sigma_i I_k - \hat{A})^{-j}\hat{B} &= C(\sigma_i I_N - A)^{-j}B \end{aligned}$$

Complexity

For dense systems these approximations require $O(N^3)$ operations due to

- linear system solving
- singular value decomposition
- eigenvalue decomposition

Not acceptable for large sparse systems

We want to use

- Projection
- sparse linear system solvers
- sparse eigenvalue solvers
- sparse matrix equation solvers, e.g., Lyapunov, Sylvester

Asymptotic Waveform Expansion

- compute the moments T_i
- construct the Padé approximation $\hat{T}(s)$ via forming and solving a system of linear equations for parameters
- numerically unreliable to form moments
- resulting linear system is typically very ill-conditioned.
- standard approach for circuit simulation before mid 90s
- completely replaced by projection-based approaches

Modal approximation via projection

Given $T(s)$, $\hat{T}(s)$ can be constructed by a projection technique : Take $Z, V \in \mathbb{C}^{N\times k}$ with $Z^T V = I_k$ such that

$$\begin{aligned} AV &= V\,diag\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} \\ Z^T A &= diag\begin{bmatrix} \alpha_1 & \cdots & \alpha_k \end{bmatrix} Z^T \end{aligned}$$

$$\begin{aligned} \hat{T}(s) &= CV(sI_k - Z^T AV)^{-1} Z^T B \\ &= \hat{C}\,diag\begin{bmatrix} \frac{1}{s-\alpha_1} & \cdots & \frac{1}{s-\alpha_1} \end{bmatrix} \hat{B} \\ &= \frac{\gamma_1}{s-\alpha_1} + \ldots + \frac{\gamma_k}{s-\alpha_k}. \end{aligned}$$

Padé, Projections, and Krylov

$\hat{T}(s) = \hat{C}(sI_k - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at $I = \{(0, 2k)\}$
$\Longleftrightarrow \hat{C}\hat{A}^{-i}\hat{B} = CA^{-i}B, \quad \forall\, 1 \le i \le 2k$

$$\begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \cdots & \hat{T}_k \\ \hat{T}_2 & \hat{T}_3 & \iddots & \hat{T}_{k+1} \\ \vdots & \iddots & \iddots & \vdots \\ \hat{T}_k & \hat{T}_{k+1} & \cdots & \hat{T}_{2k-1} \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & \cdots & T_k \\ T_2 & T_3 & \iddots & T_{k+1} \\ \vdots & \iddots & \iddots & \vdots \\ T_k & T_{k+1} & \cdots & T_{2k-1} \end{bmatrix}$$

$$\begin{bmatrix} \hat{C}\hat{A}^{-1} \\ \vdots \\ \hat{C}\hat{A}^{-k} \end{bmatrix} \hat{A} \begin{bmatrix} \hat{A}^{-1}\hat{B} & \ldots \end{bmatrix} = \begin{bmatrix} CA^{-1} \\ \vdots \\ CA^{-k} \end{bmatrix} A \begin{bmatrix} A^{-1}B & \ldots \end{bmatrix}$$

$$\hat{A} = Z^T A V$$

Projected input and output matrices

$$\begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \vdots \\ \hat{T}_k \end{bmatrix} = \begin{bmatrix} \hat{C}\hat{A}^{-1} \\ \vdots \\ \hat{C}\hat{A}^{-k} \end{bmatrix} \hat{B} = \begin{bmatrix} CA^{-1} \\ \vdots \\ CA^{-k} \end{bmatrix} B = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_k \end{bmatrix}$$

$$\hat{B} = Z^T B$$

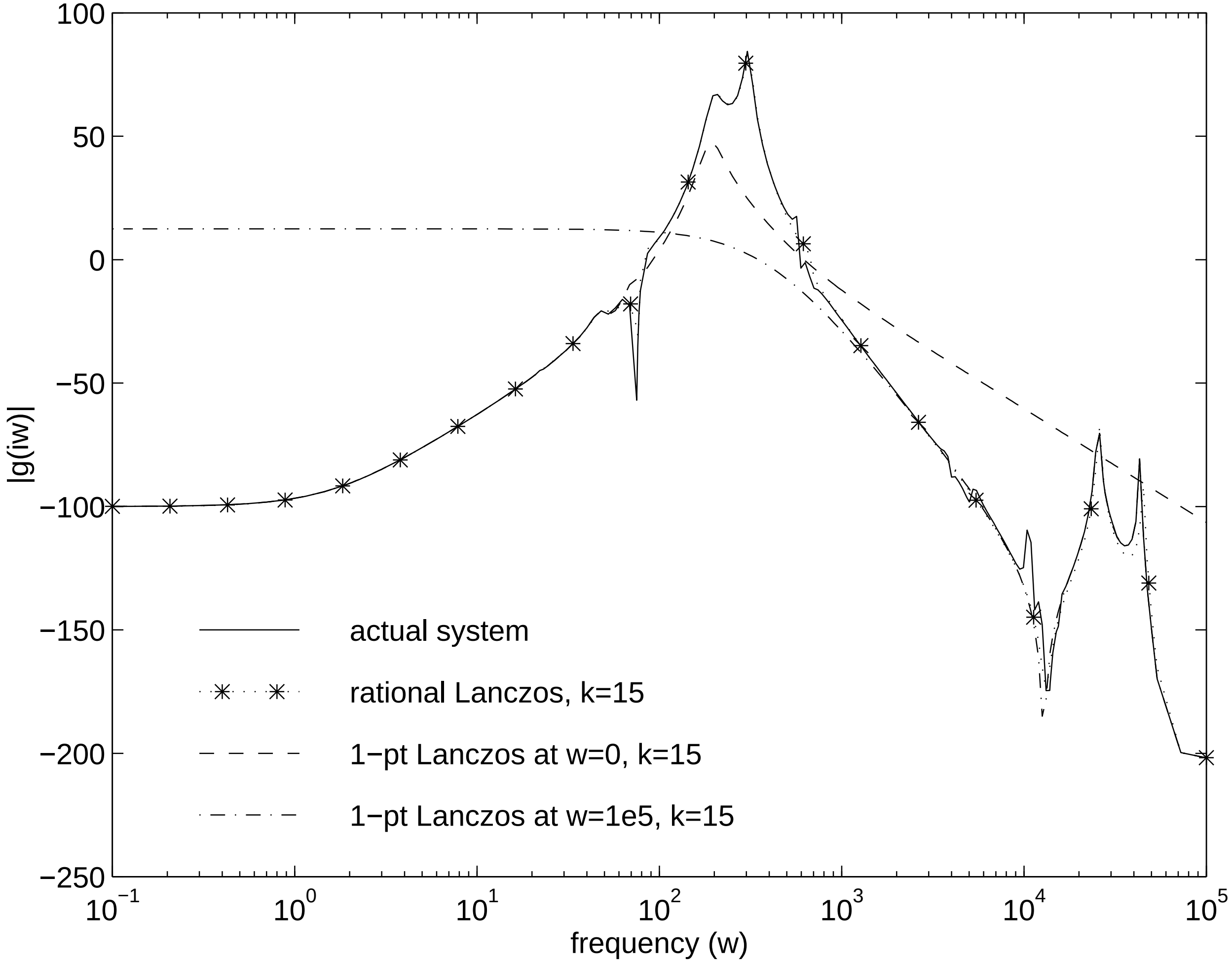
$$\begin{aligned} \hat{C} \begin{bmatrix} \hat{A}^{-1}\hat{B} & \dots & \hat{A}^{-k}\hat{B} \end{bmatrix} &= C \begin{bmatrix} A^{-1}B & \dots & A^{-k}B \end{bmatrix} \\ \hat{C} &= CV \end{aligned}$$

Rational Lanczos and Arnoldi

The projection matrices define Krylov spaces and the projected matrices can be computed efficiently via several iterative methods, e.g., Lanczos, Arnoldi etc.

$$
\begin{aligned}
\hat{A} &= Z^T A V \\
\hat{B} &= Z^T B \\
\hat{C} &= C V \\
I_k &= Z^T V \\
Im(V) &= \mathcal{K}_k(A^{-1}, A^{-1}B) \\
Im(Z) &= \mathcal{K}_k(A^{-T}, A^{-T}C^T)
\end{aligned}
$$

Multipoint is required



Brief history

Background:

- Padé approximation for model reduction;
  Shamash, etc., 1975
- Partial realization and Krylov subspaces;
  Gragg and Lindquist, 1983
- Markov and Padé approximation and Krylov subspaces; Multipoint asserted but not developed;
  Villemagne and Skelton, 1987
- Asymptotic Waveform Evaluation;
  Pillage, Rohrer, etc., 1989

Later papers use the Lanczos algorithm, a fast iterative way to form Krylov subspaces and force momement matching onto it.

Lanczos-based Model Reduction:

- Structural Dynamics: Noor-Omid, Craig, etc., 1984-1992
- Control: Boley, Kasenally, Van Dooren, etc., 1990-1994
- Circuits: Gallivan, Grimme, Van Dooren; Feldman, Freund; etc., 1994 - 1997

Unifying theory and algorithms for projection-based:

- Lanczos theory and algorithmic framework: Freund et al., 1997 - present
- Complete rational Krlov family Grimme, Gallivan, Van den Dorpe, Van Dooren, 1997-present

Balanced truncation for large scale LTI systems:

- Antoulas and Sorensen, 1999 - current

Generality of projection

- Genin, Van den Dorpe, Van Dooren and Gallivan, current

Basic Theorem of Projection

Villemagne and Skelton; Grimme, Gallivan, Van den Dorpe, Van Dooren

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}} \left((A - \sigma^{(k)} E)^{-1} E, (A - \sigma^{(k)} E)^{-1} B \right) \subseteq \mathrm{Im} X$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}} \left((A - \sigma^{(k)} E)^{-T} E^T, (A - \sigma^{(k)} E)^{-T} C^T \right) \subseteq \mathrm{Im} Y$$

$$T_i^{(j_k)} = \hat{T}_i^{(j_k)}, \quad j_k = 1, 2, \ldots, J_{(k)},$$

where $k = 1, 2, \ldots K$, $J_{(k)} \doteq J_{b_k} + J_{c_k}$ provided all of the moments exist, i.e., the matrices are nonsingular.

Including modal matching

Note only inclusion is required in theorem so other bases can be added to gain satisfy constraints

Theorem Let $\mathrm{Im} X_i$, $\mathrm{Im} Y_i$ be left and right invariant subspaces of the regular pencil $(\lambda E - A)$ with given spectrum, then the reduced order pencil $(\lambda \hat{E} - \hat{A}) \doteq Y^T(\lambda E - A)X$ has left (right) invariant subspace with the same spectrum if

$$\mathrm{Im} X_i \subseteq \mathrm{Im} X \quad \mathrm{Im} Y_i \subseteq \mathrm{Im} Y.$$

This extension allows to incorporate matching poles of the original system into the reduced order system, e.g., to stabilize it.

Approximation of DAE requires often that the reduced order system has the same algebraic conditions (these corresponds to an invariant subspace at ∞

Practicality

- Extremely flexible characterization
- Rational Power Method, Dual Rational Arnoldi, Rational Lanczos are easily defined as members of a general Rational Krylov family.
- Parallelism available at many levels
- Efficiency requires :
  - sparse approximations to linear system solutions – lose exact moment matching but Galerkin conditions hold
  - exploit shifted form of systems for preconditioning

Practicality

Point Placement and Selection:

- choose the next point at which a moment is added carefully – causes synchronization
- postprocessing can be used to get rid of some of the unnecessary basis vectors added – increases parallelism but adds cleanup operations
- clustered version of these two is natural

Practicality

- Use error estimate $\Delta(.) \doteq \hat{T}(.) - \tilde{T}(.)$
- $\hat{T}(.)$ is our current approximation and
- $\tilde{T}(.)$ is a second approximation of $T(s)$, based on adding interpolation points to that of $\hat{T}(.)$ or on interlacing points.

Point selection based on the frequency response $\Delta(\omega)$

- real points simpler than complex points and yield general trends and behavior in left half plane
- imaginary points get local information best
- mix is often best

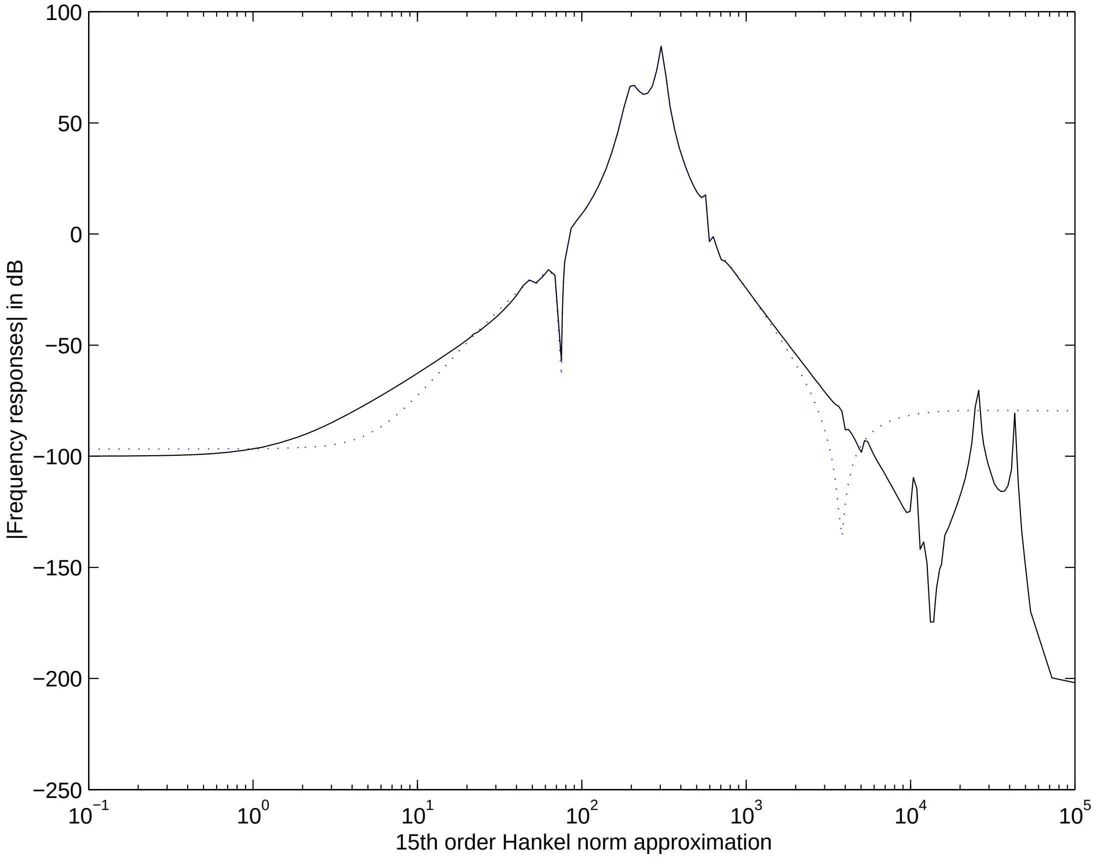
Stopping criterion is based on $\mathcal{H}_\infty$ norm of $\Delta(.)$.

Stability

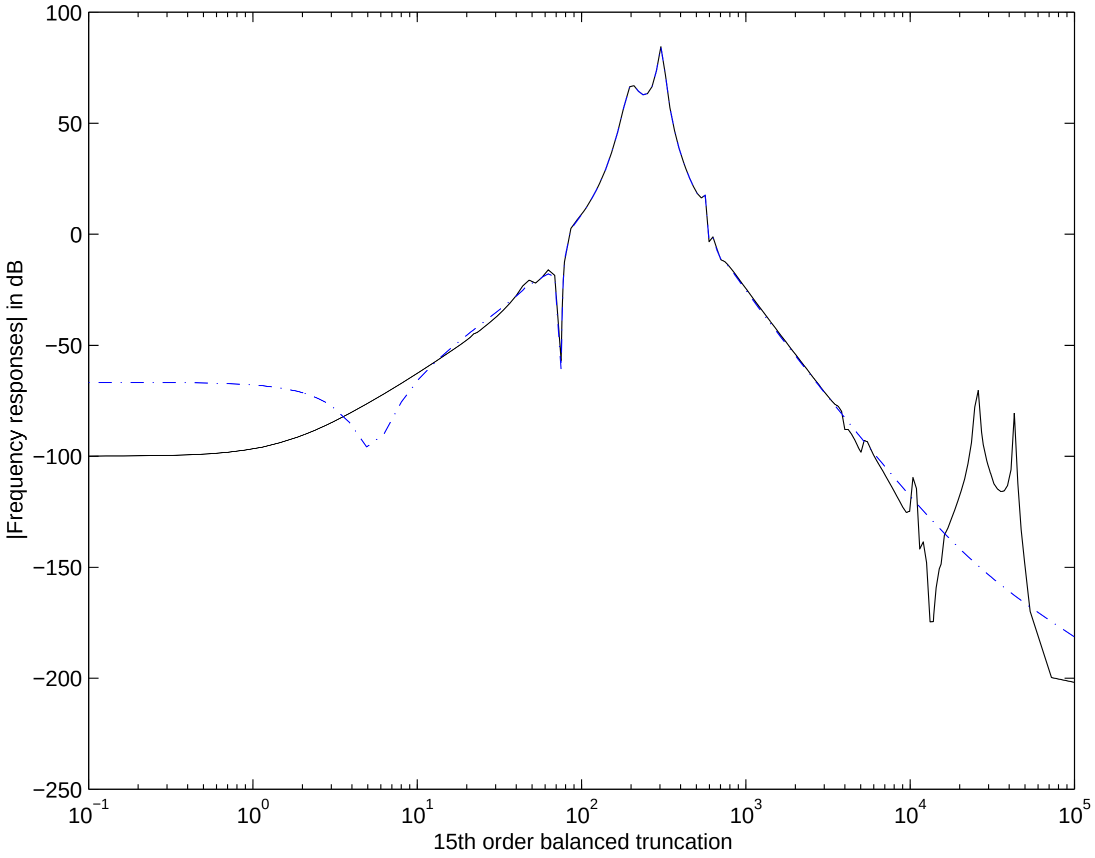
- Stability/Passivity of $\hat{T}(s)$ may require further projection on the reduced system, e.g., restarted Arnoldi (Grimme, Sorensen, Van Dooren).
- Stability/Passivity can be guaranteed when imposing only k moments to match and using the other degrees of freedom to yield a passive reduced order model (Freund,Feldman)
- Recent work by Sorensen shows that constraining the interpolation points appropriately guarantees passivity is preserved – but it does not control approximation error – a hybrid using those constraints and extra Rational Krylov steps may solve the problem.

Comparison “Optimal” and rational approximations

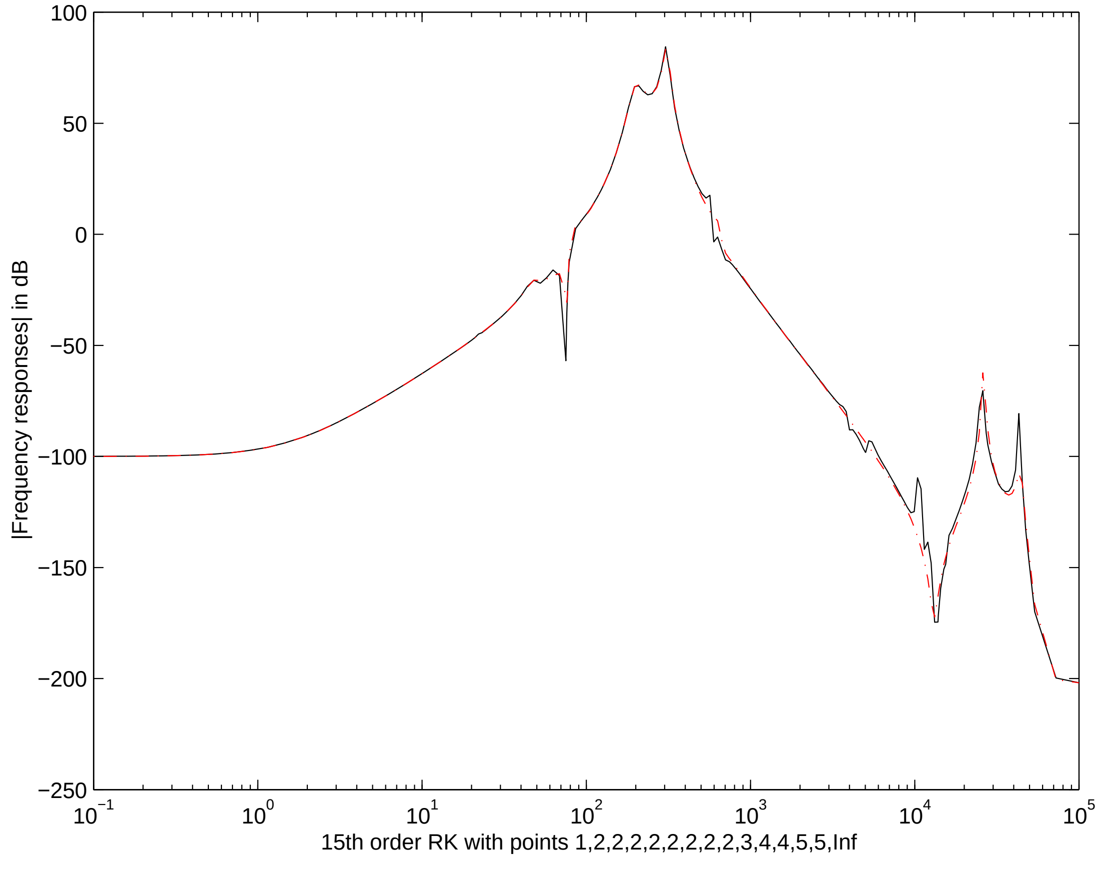
15th order approximation of 120 th order CD player



Legend : · · · Optimal Hankel norm approximation



Legend : -·-· Balanced truncation approximation



Legend : - - - Rational Krylov approximation

Approximate Balanced Truncation

- Antoulas and Sorensen
- Motivated by the fact that Hankel singular values drop off rapidly
- Avoids the need to specify interpolation conditions
- Targeted for "black box" software
- Large sparse Sylvester equation solver – ARPACK
- attempts to get at the dominant spaces of G_o and G_c
- no convergence proof
- no rigorous error bound
- works very well in practice

The basic problem

$$A^T G_o + G_o A + C^T C = 0 \quad \text{and} \quad A G_c + G_c A^T + B B^T = 0.$$

where $G_c \doteq \mathcal{C}\mathcal{C}^T$, $G_o \doteq \mathcal{O}^T\mathcal{O}$,

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & \cdots \end{bmatrix} \\ \mathcal{C} &= \begin{bmatrix} B & AB & A^2 B & \cdots \end{bmatrix} \end{aligned}$$

Given approximations V_{in} and Z_{in} to bases of the dominant spaces, $V_{up} \approx G_c V_{in}$ and $Z_{up} \approx G_o Z_{in}$ without knowing G_c or G_o.

A key kernel (Sorensen and Zhou)

Given approximations V_{in} and Z_{in} to bases of the dominant spaces, $V_{up} \approx G_c V_{in}$ and $Z_{up} \approx G_o Z_{in}$ by solving

$$\begin{aligned} AV_{up}F_r^T + EV_{up}Q_r^T + BH_r^T &= 0 \\ A^T Z_{up}F_l + E^T Z_{up}Q_l + C^T H_l &= 0 \end{aligned}$$

where F_l, Q_l and H_l are functions of Z_{in} and F_r, Q_l and H_l are functions of V_{in}

$A, E \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times m}$, $C \in \mathbb{C}^{p\times N}$, $F_{r,l}, Q_{r,l} \in \mathbb{C}^{n\times n}$, $H_r \in \mathbb{C}^{n\times m}$. and $H_l \in \mathbb{C}^{n\times p}$

A series of such projectors are produced. What are they?

Key question

We have

- Projections producing rational interpolation
- Projections producing modal approximations
- A series of Sylvester equations producing approximate balanced truncation based on approximating the Gramians

Are they related or fundamentally different approaches?

Sylvester and rational interpolation (SISO)

Suppose V and Z solve

$$
\begin{aligned}
AVF_r^T + EVQ_r^T + BH_r^T &= 0 \\
A^T ZF_l + E^T ZQ_l + C^T H_l &= 0
\end{aligned}
$$

$A, E \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times 1}$, $C \in \mathbb{C}^{1\times N}$, $F_{r,l}, Q_{r,l} \in \mathbb{C}^{n\times n}$, $H_r \in \mathbb{C}^{n\times 1}$. and $H_l \in \mathbb{C}^{n\times 1}$

Suppose also $\left[\begin{array}{c|c} sF_r - Q_r & H_r \end{array}\right]$ and $\left[\begin{array}{c|c} sF_l^T - Q_l^T & H_l^T \end{array}\right]$ are full rank for all s.

(This and SISO imply all their generalized eigenvalues have single Jordan blocks.)

Equivalence Theorem

The reduced-order system $\hat{T}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$ obtained by using projectors that have the same image V and Z is such that

$$T(s) - \hat{T}(s) = O(s + \sigma_k)^{b_k + c_k} \quad \forall\, 1 \leq k \leq K,$$

where σ_k is a generalized eigenvalue of (F_r, Q_r) and (F_l, Q_l), whose single Jordan blocks are of size b_k and c_k respectively provided $sE - A$ and $s\hat{E} - \hat{A}$ are invertible at σ_k.

Interpolation and projection (SISO)

For SISO minimal $T(s)$ of McMillan degree N and minimal $\hat{T}(s)$ of McMillan degree n projection and interpolation are equivalent!

$$\begin{cases} T(s) = C(sI_N - A)^{-1}B \\ \hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}, \quad n < N \end{cases}$$

$$\Longrightarrow \exists\, Z, V \in \mathbb{C}^{N\times n},\ Z^T V = I_n\ Z^T A V = \hat{A},\ Z^T B = \hat{B},\ CV = \hat{C}.$$

Sketch of the proof :

If Mc Millan degree $E(s) \geq 2n \Longrightarrow$ Multipoint Padé,

Otherwise $\Longrightarrow$ common poles $\Longrightarrow$ Multipoint Padé + Modal Truncation.

Consequence : $\hat{T}(s)$ interpolates $T(s) \not\Rightarrow \hat{T}(s)$ is a good choice!

There is always a Sylvester equation!

- Part $(F_1, 0)$: $AV_1 + V_1 F_1 = 0 \Longrightarrow$ Modal Approximation
- Part (F_i, G_i) observable (each F_i is one Jordan block):

$$AV_i + V_i \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} + B \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad Im(V_i) = \mathcal{K}_k \left((A + \lambda I)^{-1}, (A + \lambda I)^{-1} B \right)$$

$$\Rrightarrow \quad \begin{cases} Observable\ part\ of\ (F, G) \Longleftrightarrow Interpolation \\ Non\ Obs.\ part\ of\ (F, G) \Longleftrightarrow Modal\ Approximation \end{cases}$$

SISO Consequences

- The series of Sylvester equations in the approximate balanced truncation produce a series of rational interpolation approximations
- There the interpolation conditions are implicitly specified by controlling the choice of F_r, Q_r F_l and Q_l!
- Controlling the choice of F_r, Q_r F_l and Q_l by specifying the spectra in Jordan form yields a Rational Krylov method directly from the recursion to solve the Sylvester equation!!
- Sylvester equations and the shift-and-invert based Rational Krylov algorithms are two algorithmic approaches that produce the same reduced order models.

What about MIMO? – Check Sylvester

Let $V \doteq \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ and $y_i \in \mathbb{C}^{m\times 1}$ with all Jordan blocks of size 1.

$$AV + V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} + B \begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix} = 0$$

Choose Z so $Z^T V = I_n$ and construct $\hat{T}(s)$

It follows that for any $1 \le i \le n$

$$\begin{aligned} v_i &= (\lambda_i I_n - A)^{-1} B y_i \\ T(\lambda_i) y_i &= \hat{T}(\lambda_i) y_i \end{aligned}$$

- Rational Interpolation : match $T^{(j)}(\sigma) = \hat{T}^{(j)}(\sigma)\ j = 1, \cdots, k_\sigma$
- Left Tangential Interpolation :

$$\begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} T(\sigma) = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} \hat{T}(\sigma)$$

$$\iff \quad \begin{cases} \xi_1 t_{11}(\sigma) + \xi_2 t_{21}(\sigma) = \xi_1 \hat{t}_{11}(\sigma) + \xi_2 \hat{t}_{21}(\sigma) \\ \xi_1 t_{12}(\sigma) + \xi_2 t_{22}(\sigma) = \xi_1 \hat{t}_{12}(\sigma) + \xi_2 \hat{t}_{22}(\sigma) \end{cases}$$

- Right Tangential Interpolation :

$$T(\sigma) \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \hat{T}(\sigma) \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

- Two-Sided Tangential Interpolation combines both

Tangential Interpolation

If the range of

$$\left[\begin{array}{ccc}(\lambda I - A)^{-1}B & \cdots & (\lambda I - A)^{-k}B\end{array}\right]\left[\begin{array}{ccc}\eta_0 & \cdots & \eta_{k-1} \\ & \ddots & \vdots \\ & & \eta_0\end{array}\right]$$

is in $Im(V)$ then the vector polynomial $y(s) = \sum_{i=0}^{k-1} \eta_i (s-\lambda)^i$ satisfies a right tangential interpolation condition

$$(T(s) - \hat{T}(s))y(s) = O(s-\lambda)^k$$

(essentially this requires a nontrivial Jordan block for λ)

Left tangential interpolation conditions can be derived similarly.

Tangential interpolation and projection/Sylvester

- Tangential interpolation via projectors produced from a Sylvester equation is a powerful MIMO model reduction tool.
- It may be the key that allows a rigorous understanding of approximate balanced truncation and rational Krylov-based approaches for MIMO and facilitate their unification.
- It is possible to show that it is not always possible to find projectors that produce $\hat{T}(s)$ from $T(s)$ in the case of MIMO systems – key restrictions present for SISO are lost. So tangential interpolation and projection are not universal for MIMO.
- However, Van den Dorpe and Genin have conjectured that any $\hat{T}(s)$ of degree n can be produced from any $T(s)$ of degree N via tangential interpolation defined projectors for practical situations in large-scale model reduction, i.e., when $N - n$ is sufficiently large.

Conclusions for equation-based model reduction

- We have successfully transferred techniques from the linear algebra and control communities into the circuit simulation community – the Rational Krylov approaches are now standard.
- projection-based approaches provide a flexible family of model reduction methods
- We have extended, clarified and unified the theoretical understanding of the problem in a way that includes algorithmic and implementation insight, i.e., our theory guides our efficient algorithm design.
- We have aided in the dispersal of the updated knowledge and algorithms to other applications, e.g., electromagnetics application by Michielssen and Weile.

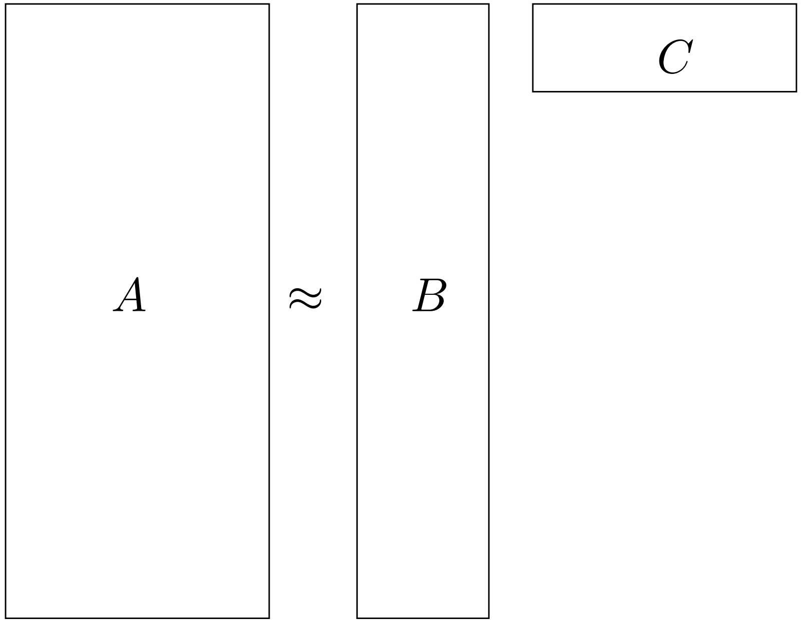
Conclusions for equation-based model reduction

- The recent advances in understanding the generality and links to other approaches promise a new round of advances from an algorithmic and library point of view.
- Progress is needed on the specification of the interpolation conditions required to satisfy **both** error requirements and property preservation
- Combined implicit and explicit specification strategies that use Sylvester and Rational Krylov methods look very promising.
- We are currently thinking about how to apply this to model reduction in a closed-loop control setting and a linear time-varying setting

Incremental tracking of dominant singular spaces

Given $A_{m\times n}$, approximate by a rank k factorization $B_{m\times k} * C_{k\times n}$

$$\min \|A - BC\|_2, \quad k \ll m, n$$

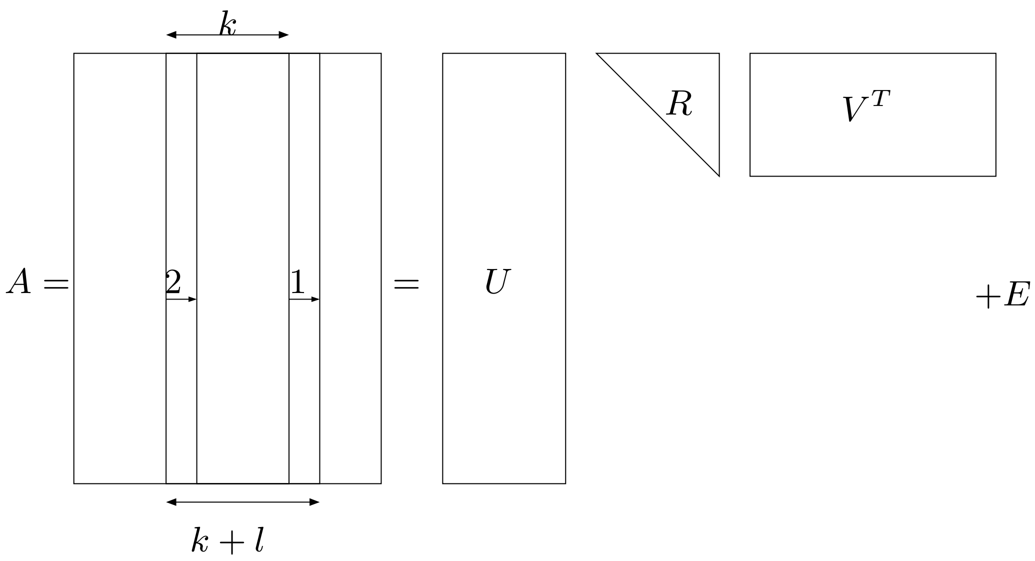


Applications in :

- Image compression
- Information retrieval
- Image recognition
- Model reduction (P.O.D)

Idea: Windowing

Track dominant spaces with a sequence of windows SVD's of dimension $m \times (l+k)$



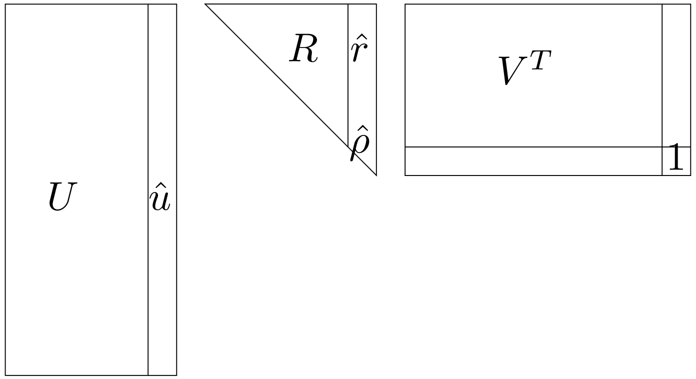
1: expand by appending l columns (Gram Schmidt)

2: contract by deleting l columns (SVD update)

<u>Total cost</u>: $10mnk$ (Givens) or $8mnk$ (Householder) instead of $O(mn^2)$

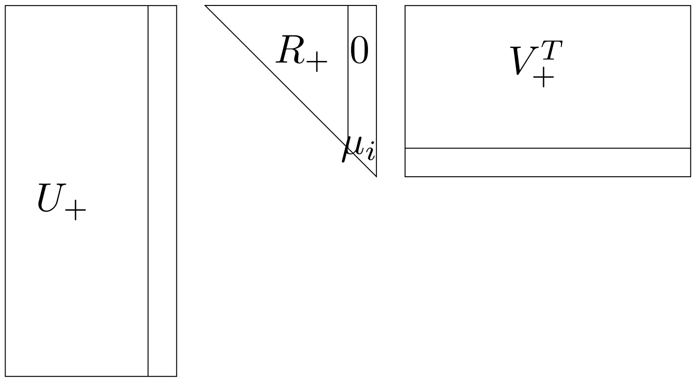
Details (for $l = 1$) of step i, $i = k+1, \ldots, n$

<u>Expand</u>: appending column a_+ (Gram Schmidt): $4mk$ flops



$= \hat{U}\hat{R}\hat{V}^T$

<u>Downdate</u>: removing smallest singular value of $\hat{R}$: $6mk$ flops



$= (\hat{U}G_u) * (G_u^T \hat{R} G_v) * (G_v^T \hat{V}^T)$

$$
P_3 P_2 P_1 \begin{pmatrix} \rho & \rho & \rho & \rho & \eta_1 \\ \phi_1 & \rho & \rho & \rho & \eta_2 \\ 0 & \phi_2 & \rho & \rho & \eta_3 \\ 0 & 0 & \phi_3 & \rho & \eta_4 \end{pmatrix} Z_1 Z_2 Z_3 = \begin{pmatrix} \rho & \rho & \rho & 0 & 0 \\ 0 & \rho & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 1 \end{pmatrix}
$$

The elimination and fill-in structure for the two-sided algorithm with $k = 3$. (Z_i eliminates η_i

- An evolution equation

$$\dot{x} = F(x)$$

is replaced by a reduced order equation

$$\dot{a} = U_k^H F(U_k a) = f(a)$$

- information can be recovered by integrating the reduced order equation rather than interpolating between saved states (columns of A)
- If V_k^T is tracked then $U_k \Sigma_k V_k^T e_i \approx x_i$ can be used instead of integration.
- The form of the differential equation influences the cost of the production of the reduced order system and whether or not moving between the reduced state space and the original state space can be avoided

Perturbation Theorem The recursive algorithm produces "approximate" matrices $\bar{V}_{(i)}, \bar{Q}_{(i)}$ and $\bar{R}_{(i)}$ that satisfy *exactly* the perturbed equation

$$[A(:, 1\ : i) + E]\bar{V}_{(i)} = \bar{Q}_{(i)}\bar{R}_{(i)}, \quad (\bar{V}_{(i)} + F)^T(\bar{V}_{(i)} + F) = I_k,$$

with the bounds (up to $O(u^2)$ terms) :

$$\|E\|_F \leq \epsilon_e \|A\|_2, \ \ \epsilon_e \leq 26k^{3/2}nu, \quad \|F\|_F \leq \epsilon_f \leq 9k^{3/2}nu.$$

and in practice

$$\epsilon_e \leq 26k^2 u, \quad \epsilon_f \leq 9k^2 u.$$

Note these bounds do *not* depend on m, the largest dimension of A

Orthogonality Theorem Let (a given matrix) $\bar{V} \in \mathcal{R}^{n\times k}$ "select" k columns of the matrix $A \in \mathcal{R}^{m\times n}$, and let

$$A\bar{V} = QR, \quad Q^T Q = I_k,$$

with R upper triangular, be its exact QR factorization. Let

$$A\bar{V} + G = \bar{Q}\bar{R}, \quad \|G\|_F = \epsilon_g \|A\|_2 \approx u\|A\|_2, \tag{1}$$

be a "computed" version, where $\bar{Q} = Q + \Delta_Q$, $\bar{R} = R + \Delta_R$. Then under a mild assumption, we can bound the loss of orthogonality in $\bar{Q}$ as follows:

$$\|\bar{Q}^T\bar{Q} - I_k\|_F \leq \sqrt{2}\epsilon_g \kappa_2(R)\kappa_R(A\bar{V}) \leq 2\epsilon_g \kappa_2^2(R), \quad \epsilon_g \approx u.$$

Estimate quality of approximation

Quality of rank k approximation $\hat{A} = \hat{U}\hat{\Sigma}\hat{V}^T$ of $A = U\Sigma V^T$ estimated by :

Canonical angles :

$$\cos\theta_k \doteq \|U^T(:,k)\hat{U}\|_2 \qquad , \qquad \cos\varphi_k \doteq \|V^T(:,k)\hat{V}\|_2$$

$$\tan\theta_k \preceq \tan\hat{\theta}_k \doteq \frac{\hat{\mu}^2}{(\hat{\sigma}_k^2 - \hat{\mu}^2)} \qquad , \qquad \tan\varphi_k \preceq \tan\hat{\varphi}_k \doteq \frac{\hat{\mu}\hat{\sigma}_1}{(\hat{\sigma}_k^2 - \hat{\mu}^2)}$$

Singular values :

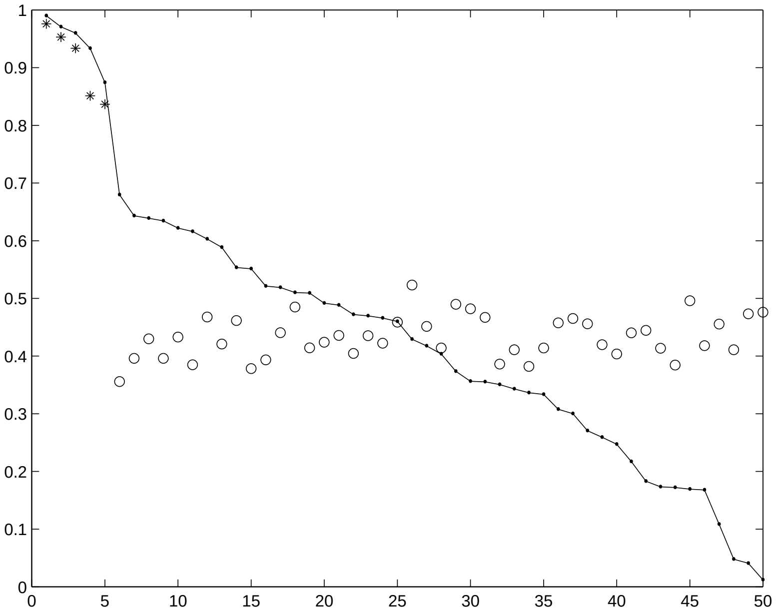
$$|\hat{\sigma}_i - \sigma_i| \approx \frac{\hat{\mu}^2}{2\hat{\sigma}_i}$$

Approximation error :

$$\|E\|_2 \doteq \mu \geq \sigma_{k+1}$$

$$\hat{\mu} = \max|\mu_i| \approx \mu \qquad \text{(true error)}$$

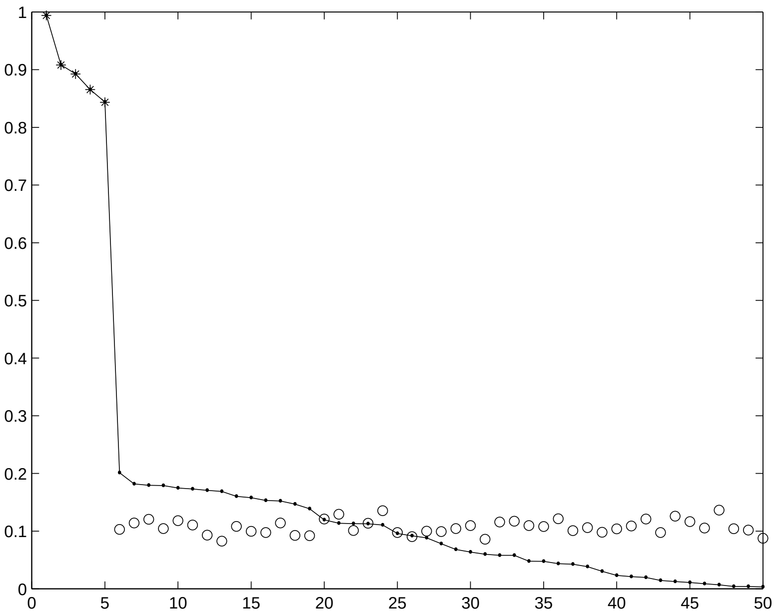
Gap γ : 0.19458 , $\sigma_{k+1} = 0.67978$



—— true sv's $\sigma_i(A)$, $*$ approximated sv's $\hat{\sigma}_1^{(n)}, \ldots, \hat{\sigma}_k^{(n)}$, $\circ$ dismissed sv's $\mu_{k+1}, \ldots \mu_n < \sigma_{k+1}$

$\sigma_1 = 0.99008$	$\hat{\sigma}_1 = 0.97613$
$\sigma_2 = 0.97084$	$\hat{\sigma}_2 = 0.95301$
$\sigma_3 = 0.96010$	$\hat{\sigma}_3 = 0.93379$
$\sigma_4 = 0.93338$	$\hat{\sigma}_4 = 0.85142$
$\sigma_5 = 0.87437$	$\hat{\sigma}_5 = 0.83675$
$\mu = 0.73768$	$\hat{\mu} = 0.52330$
$\cos\theta_k = 0.93000$	$\cos\hat{\theta}_k = 0.82233$
$\cos\phi_k = 0.83881$	$\cos\hat{\phi}_k = 0.71038$

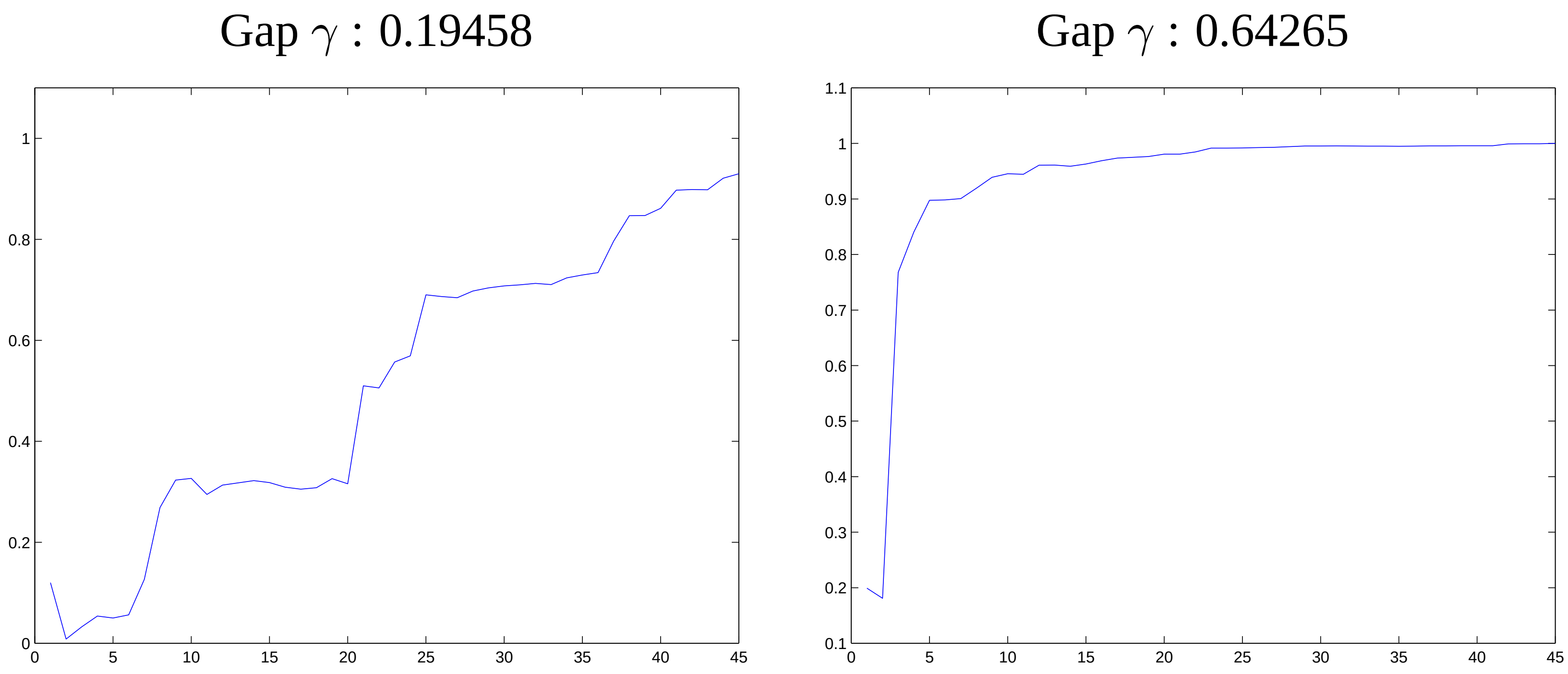
Gap $\gamma : 0.64265$, $\sigma_{k+1} = 0.20121$



—— true sv's $\sigma_i(A)$, $*$ approximated sv's $\hat{\sigma}_1^{(n)}, \ldots, \hat{\sigma}_k^{(n)}$, $\circ$ dismissed sv's $\mu_{k+1}, \ldots \mu_n < \sigma_{k+1}$

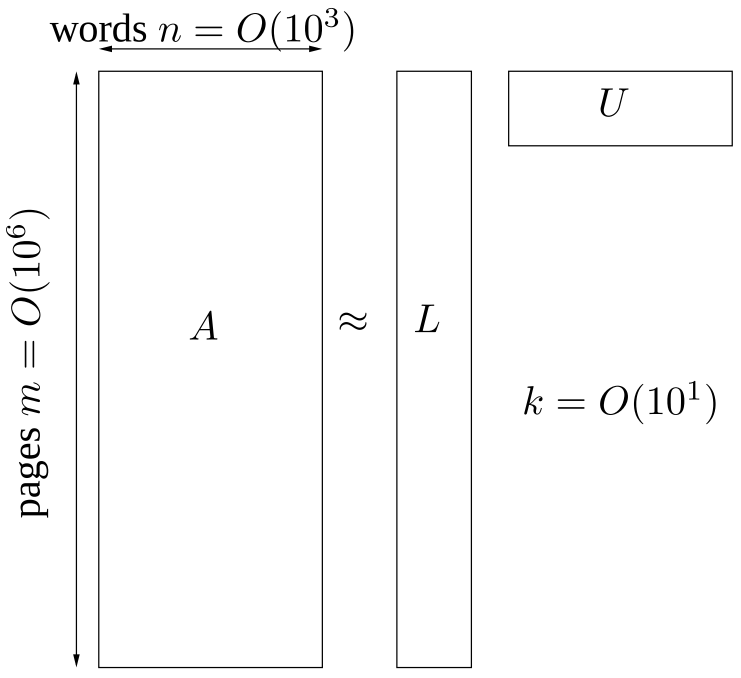
$\sigma_1 = 0.99430$	$\hat{\sigma}_1 = 0.99418$
$\sigma_2 = 0.90840$	$\hat{\sigma}_2 = 0.90815$
$\sigma_3 = 0.89284$	$\hat{\sigma}_3 = 0.89250$
$\sigma_4 = 0.86560$	$\hat{\sigma}_4 = 0.86551$
$\sigma_5 = 0.84387$	$\hat{\sigma}_5 = 0.84357$
$\mu = 0.20140$	$\hat{\mu} = 0.13631$
$\cos\theta_k = 0.99998$	$\cos\hat{\theta}_k = 0.99459$
$\cos\phi_k = 0.99935$	$\cos\hat{\phi}_k = 0.94334$

How quickly do we track the subspaces?



How $\cos\theta_k^{(i)}$ evolves with the time step i

Information retrieval



Low rank approximation useful for:

- Low memory requirement $O(k(m+n))$
- Fast queries $Ax \approx L(Ux)$ in $O(k(m+n))$ time
- Approximation obtained in $O(kmn)$ time using windowing

Image sequences

Each column of A is one image

Original : $m = 28341, n = 100$ Approximation : $k = 6$

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Related Work

- *Efficient Arrowhead* - Chandrasekaran, Manjunath, Wang, Winkeler, and Zhang, Graphical Models and Image Processing *1997*.
- *Sequential Karhunen-Loeve* (*SKL*) - Levy and Lindenbaum, *2000*, IEEE Trans. Image Processing .
- *Incremental SVD* (*IncSVD*) - Chahlaoui, Gallivan, and Van Dooren,*2000* CIRO 2000, to appear SIMAX.

Complexity

Algorithm	One-sided	Two-sided
SKL	$2mn\frac{k^2+3kl+2l^2}{l}$	$2mn\frac{k^2+3kl+2l^2}{l} + \frac{n^2k^2+n^2kl}{l}$
IncSVD (G)	$10mnk + 4mnl$	$10mnk + 4mnl + 3n^2k$
IncSVD (HH)	$8mnk + 4mnl$	$8mnk + 4mnl + \frac{n^2k^2+n^2kl}{l}$

Currrent and Future Work

- Finalizing empirical and analytical investigation of dynamic block size and loss of orthogonality
- Beginning study of PODs to represent perturbed trajectories when analyzing trajectories of differential equations with stochastic parameters.
- Beginning algorithm/architecture interaction study for incremental tracking on latest graphics attached processors for problems similar to those address by SKL

Searching on the Grassmann manifold for projectors

- Collection of images $D \in \mathbb{R}^n$ grouped into classes a priori
- Training data selected by human to represent essential characteristics of each class
- Seeking a subspace with dimension d with $U \in \mathbb{R}^d$ to encode the images $U^T D = C$.
- Standard bases often depend upon statistical assumptions that may not be accurate enough for given data and that may have nothing to do with the metric for recognition
- We search the Grassmann manifold via a stochastic gradient method to optimize the recognition metric on the test data
- We assume that substantial amounts of off-line preprocessing are allowed in order to improve on-line recognition.

Motion on the Grassmann manifold

Assume you have a basis S_0 for a space $\mathcal{S}$, and you wish to move along the Grassmann manifold.

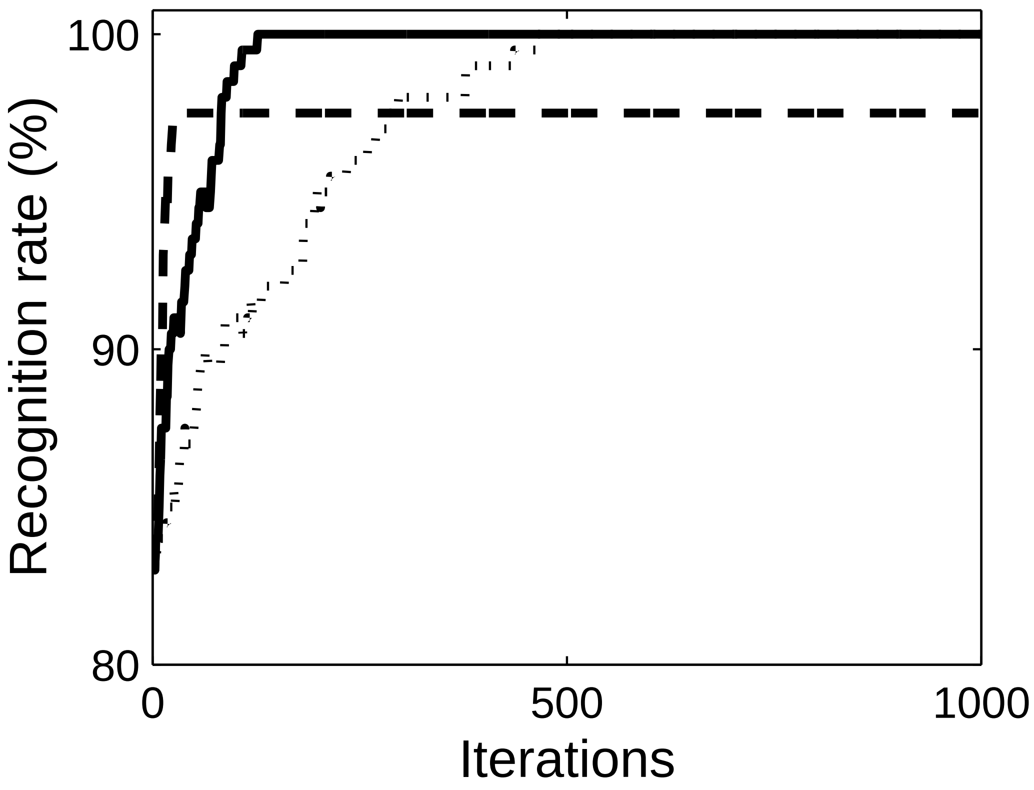
The trajectory can be given in terms of bases but only their spaces are of interest.

$$S(t) = Qe^{\begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix}} Q^T S_0 \quad Q^T S_0 = \begin{bmatrix} I_d \\ 0 \end{bmatrix}$$

$B \in \mathbb{R}^{n-d \times d}$ are the directional velocities and represent the degrees of freedom of motion on the manifold from a given point.

This can be evaluated in $O(nd^2)$ for each time point.

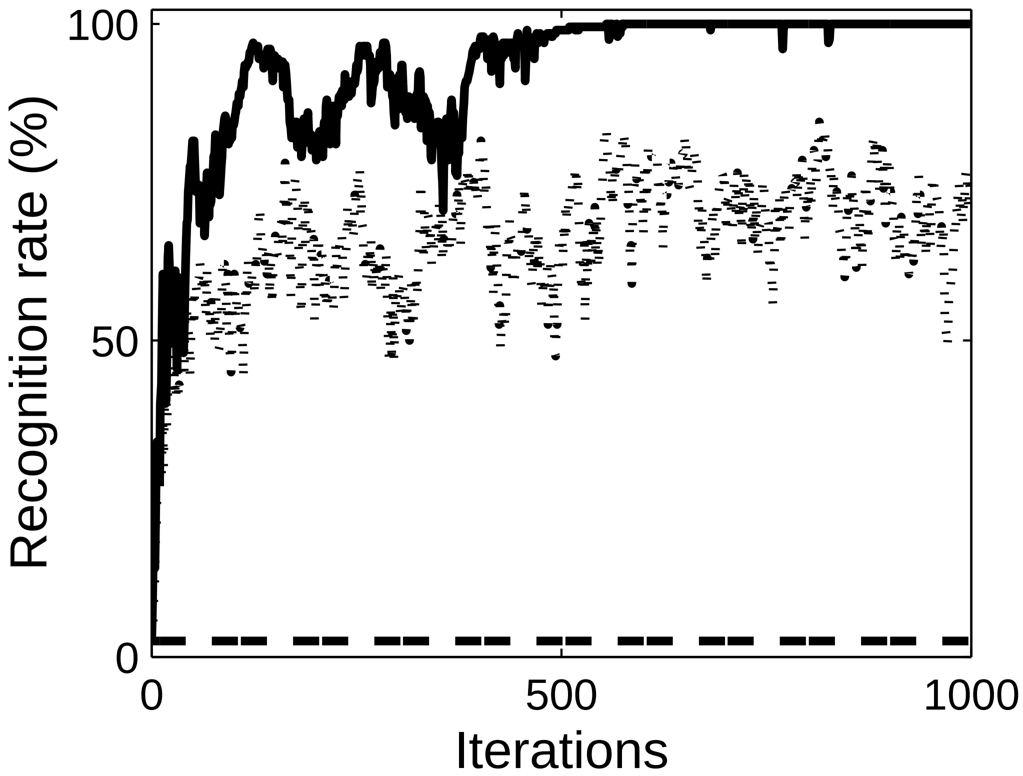
Recognition versus t



$n = 10304$, (that is 92×112), $d = 5$, $k_{train} = 5$, and $k_{test} = 5$.

$X_0 = U_{PCA}$

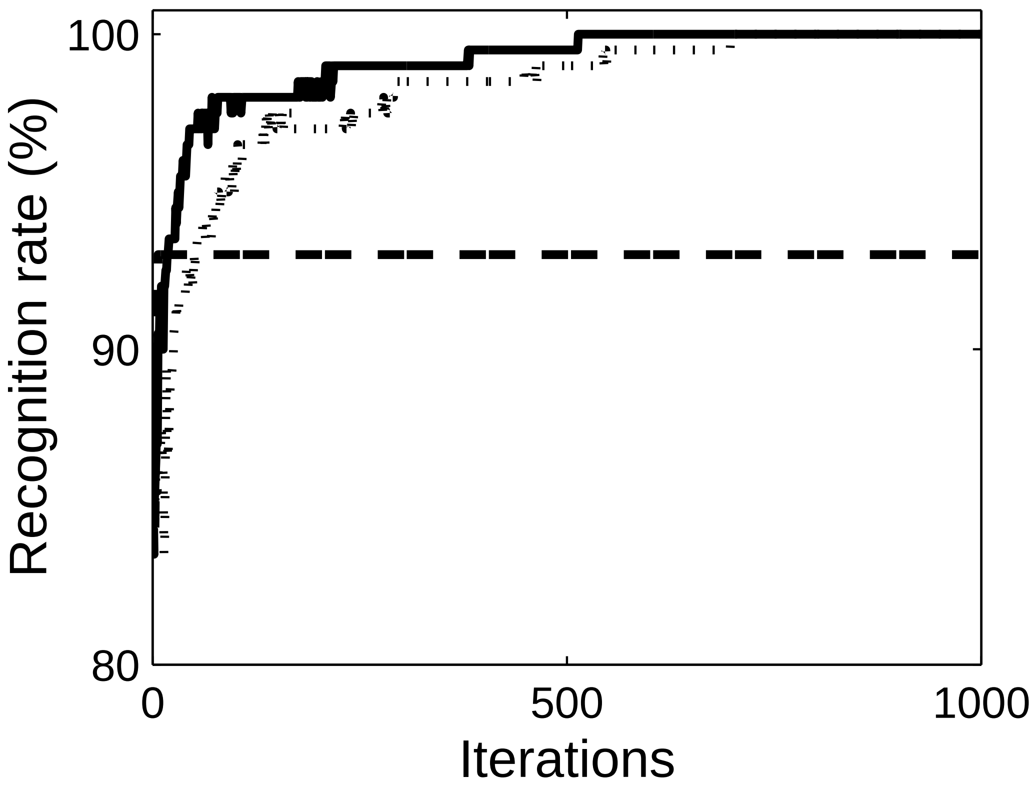
Recognition versus t



$n = 10304$, (that is 92×112), $d = 5$, $k_{train} = 5$, and $k_{test} = 5$.

$X_0 = U_{ICA}$

Recognition versus t



$n = 10304$, (that is 92×112), $d = 5$, $k_{train} = 5$, and $k_{test} = 5$.
$X_0 = U_{FDA}$.