To the most precious women in my life:
my mother, Ljudov’ Sergeevna
my wife, Svetlana
and
my daughter, Polina
Acknowledgments

I would like to thank Stanley R. Pliska for introducing to this topic, helping me with comments and suggestions, and for his patience and encouragement. I would like to express my gratitude to Susan Friedlander for her valuable help during all aspects of my graduate education. I would also like to thank Charles Tier and Floyd Hanson for discussions, comments and their help at various levels of my research project. Finally, I would like to thank Dr. Alexander Lipton-Lifschitz from Credit Suisse for taking time out from his busy schedule to serve as my external committee member.

I would like to especially thank my wife, Svetlana, for all her love, understanding and never ending patience with me. I am grateful to my mother, Ljubov Sergeevna, and my step-father, Nikolaj Trofimovich, without whose help I would not have been able to complete my graduate studies and to finish my thesis. A very special thanks goes to Polina for being such an adorable daughter.

In conclusion, I recognize the financial assistance (Teaching Assistantship) of the Department of Mathematics, Statistics and Computer Science at the University of Illinois at Chicago.

Chicago, March 2003

Yevgeny Goncharov
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Summary

The models in the literature for valuation of mortgage contracts subject to prepayment can generally be classified into one of two categories: the option-based models and the empirical mortgage-rate-based models. Using risk-neutral martingale methods together with the intensity-based approach borrowed from credit risk modeling, this thesis develops a framework that not only generalizes but considerably extends both approaches and develops new ones. The analysis of prepayment intensity functions and borrower’s refinancing incentives advances our understanding of mortgage securities and can make a significant contribution to better prediction of prepayment rates. In particular, we show several reasons why the modern option-based models imply large transaction costs and very slow-to-prepay borrowers. Our general model is not tied to a particular numerical procedure as option-based mortgage models in the literature. As an example we show that Stanton’s model [43] is in fact just a variant of a splitting-up, numerical method of the first order applied to our model. Throughout the thesis we point out various new ways to develop mortgage modeling.
Chapter 1

Introduction

As when, O lady mine!
With chiselled touch
The stone unhewn and cold
Becomes a living mould.
The more the marble wastes,
The more the statue grows.

Michelangelo Buonarroti
(1474-1564)

Mortgage markets, both primary and secondary, evolved considerably during the last two decades. A large body of literature, both academic and institutional, is devoted to the rational pricing of mortgage-related assets. Considerable progress has been made with the understanding of relevant factors for mortgage pricing, but more work is required for a fully satisfactory, complete understanding.

The prepayment option is what makes mortgage related securities complicated assets to price. Mortgagors have the option to prepay fully or partially their loans prior to the maturity dates. This right has a dramatic effect on valuation by introducing cash flow uncertainty which depends on the mortgage holder’s view of possible future opportunities (e.g., the mortgage holder’s expectation of the future behavior of the yield curve) to refinance the loan.

A very important feature of a mortgage model is how one models prepayment incentives of the borrower. This determines the borrower’s decision making implied by the model. Due to the different characteristics of approaches to model prepayment incentive measures (i.e., a borrower’s idea of profitability of prepayment), mortgage models in the literature can be classified into two groups. First group is a type of model which is closely related to pricing of American options and therefore is called the option-based or option-theoretic approach. The most advanced in the sense of model “sophistication” so far, this measure of prepayment incentive is completely endogenous. The central object of the approach is a borrower’s liability, which can be defined as the present value of cash flow that the borrower will pay off to repay his/her loan plus various transaction costs which are incurred in case of prepayment or default.\footnote{Some researchers, e.g., Stanton \cite{43}, understand transaction costs in a wide sense: it includes monetary as well as psychological costs, e.g., inconvenience associated with finding a bank, filling out forms, spending time, bad credit in the case of default, etc.}

Option-based models measure the mortgagor’s incentive to prepay as the difference between this liability and outstanding principal.\footnote{If the possibility of default is also taken under consideration, then the option-based models measure the mortgagor’s incentive to default as the difference between this liability and the price of underlying real estate.} This assumption somewhat simplifies the problem of mortgage modelling, because it removes the necessity of endogenous mortgage rate modelling, although it implies the
CHAPTER 1. INTRODUCTION

The option-based approach has evolved from frictionless models with optimal prepayment behavior (e.g., Dunn and McConnell [13], Kau, Keenan, Muller and Epperson [32]), where the mortgage liability to the borrower and mortgage asset to the investor are not distinguished (or they differ due to fees only) and the borrowers terminate their mortgages if and only if it is financially optimal, to models which recognize the importance of taking into account substantial presence of transaction costs and non-optimal behavior (the borrower can fail to prepay when it is financially profitable and he/she can prepay when it is not profitable to do so) as well as heterogeneity of borrower characteristics. Transaction costs were first incorporated in the model as a part of refinancing threshold for optimal liability (e.g., Dunn and Spatt [14]). Then Johnson and Van Drunen [26], among others, extended the model to allow refinancing costs that vary across borrowers to capture burnout effect. All these mentioned models assumed optimal prepayment plus “background” prepayment due to relocation, divorce, etc. Stanton [43] first acknowledged the fact that borrowers fail to prepay even if it is optimal to do so. McConnell and Singh [36] agreed with Stanton on this point but still based their model on optimal frictionless liability while Stanton based his model on a “correct” value of liability. The reason for calling Stanton’s liability “correct” is the following. If one thinks about refinancing his/her mortgage, then he/she should consider the loss of the prospect of future prepayment which, in turn, is not optimal. Put another way, when he/she thinks that it may be worthwhile to postpone prepayment, then he/she should also consider the possibility of failing to use that future opportunity too. Stanton [43] included this possibility in his definition of the borrower’s liability while McConnel and Singh [36] ignored it. The other already mentioned models did not consider non-optimality of financially profitable decisions at all.

The second class of models are sometimes called “option-based” models too (e.g., Deng and Quigley [8] and Deng, Quigley and Van Order [9]), though the name does not precisely (if we understand “option approach” in the sense of Merton [37]) reflect the nature of prepayment decision making that is assumed in the models. To avoid confusion we will call them mortgage-rate-based models or MRB for short. In this type of model the prepayment policy is based on a comparison of prevailing mortgage and contract rates. This comparison can be done with a “naive” mortgage formula (see formula (2.1) below) such as a precise calculation of how much is saved on, say, monthly payments by refinancing (e.g., Deng [7], Deng and Quigley [8], Deng, Quigley and Van Order [9]), or it can be done through a direct comparison such as the difference or ratio of two mortgage rates (e.g., Schwartz and Torous [41, 42], Richard and Roll [40], Kariya and Kobayashi [28], Kariya, Pliska and Ushiyama [27]). This borrower’s decision making assumption relates to the case where borrowers can refinance only to the same type of mortgage (as opposed to the already mentioned implied freedom of choice and absence of borrower’s preferences with the option-based approach). To the best of this author’s knowledge, nobody in this context used or even defined endogenous mortgage rates (see section 2.3 of this thesis for work in this direction). Because of exogenously (empirically) defined mortgage rates, these reduced models are often called empirical models (the model of Schwartz and Torous [41], for example, is called “purely empirical” by Stanton [43]). Usually, the 10 year Treasury yield is used as a proxy for the mortgage rate. Early MRB models (e.g., Schwartz and Torous [41]) recognized non-optimality of the borrower’s behavior; later models incorporated borrower heterogeneity (e.g., Deng, Quigley and Van Order [9]) and transaction costs as well (e.g., Kariya, Pliska and Ushiyama [27], where the authors assumed optimal prepayment however).

3For example, the mortgage holder prepays as soon as the mortgage rate drops below a specified threshold.
4Or to some finite set of mortgage types (for example, 15- and 30-year fixed rate mortgages), if the approach is generalized (see note on the extension to multi-mortgage market in Chapter 3).
A common feature of the literature on the option-based approach with non-optimal prepayment incentive measures (i.e., the possibility of non-optimal refinancing is included in the liability definition) is the way the dependence of prepayment behavior on future interest rates is handled. This is in the spirit of traditional backward binomial pricing, the illustrative and intuitive method often used to price American options. The option-based researchers discretize the termination process in time and then, at each time step, they consider the probabilities of prepayment and/or default. There are no attempts to find a continuous underlying process and/or its corresponding differential equation in the case when all factors are modeled as diffusion processes. Accurate knowledge of the true underlying process is important since it allows one to employ powerful high-order numerical methods to solve the problem. At the same time the only advantage of a binomial model is its simplicity. If other processes (such as an interest rate process) are simulated by a binomial tree too (see Kau, Hilliard and Shawson [29]), then one should be careful with the state space mesh because of the conditional stability of this explicit numerical algorithm. It is a well known fact of numerical analysis that the time step must be quadratically smaller than step sizes of the other variables in explicit approximations of diffusion processes. Therefore, if one wants to represent the state space accurately, the researcher must consider a small time step size, which is often undesirable for such a long term contract as a mortgage.

Stanton [43] and Downing, Stanton and Wallace [11] expressed the interest rate and the real estate (in the latter paper) part of the mortgage process with continuous diffusions and left only the borrower’s decision making as discrete. This allowed them to employ a stable numerical procedure for the continuous part. While this procedure guarantees stability, at the same time, as will be shown in the present paper, the explicit method which they employed for the borrower’s decision making makes the model a first order approximation of the “underlying” continuous-time model, thereby reducing the efficiency of higher order numerical schemes for the continuous part (Crank-Nicolson in Stanton [43], parallel hopscotch in Downing, Stanton and Wallace [11]).

Meanwhile, there is a large credit-risk literature for securities that are subject to default. Much of this literature is concerned with an intensity-based approach and continuous time models. General models are created which can account for all possible default specifications (including possible occurrence of default at discrete dates). Some research has been done on default risk premia and the relationship between the real world default probability and the risk-neutral probability measure, which is known to be a corner stone of pricing theory. Still, in spite of some successful mathematical theory for credit risk modelling, nobody has tried to use related results for mortgage modelling so far. But after all, from a mathematical point of view nothing precludes one from interpreting prepayment as a “default” in the intensity-based approach to pricing credit risk.

In this thesis we use the intensity-based approach and risk-neutral, martingale methods to define a mortgage model in a rigorous way and derive formulae that are convenient for calculations. The formulae are valid for different types of mortgages, including adjustable rate and all kinds of fixed rate mortgages (such as graduated payment, growing equity or the most popular level-payment mortgages). Both the option-based and MRB approaches are special cases of our intensity-based approach. Moreover, in the case where all underlying factors are diffusions, we show that the main equation of the option-based approach is a semi-linear reaction-diffusion PDE.\footnote{It is done for optimal prepayment liability only. For example, Dunn and McConnell [13] presented a free boundary value problem for liability. Not surprisingly, this is exactly the problem of American options (optimal prepayment) as opposed to the solution of quasi-linear PDE’s with fixed boundaries in the case of non-optimal prepayment liability (as we will show in the present paper).}

\footnote{The topic of the risk-neutral probability measure is ignored with respect to prepayment probabilities in the existing literature on mortgage modelling.}

\footnote{In a more general setting which has not been considered in the literature yet, the PDE is quasi-linear. See a note to section 2.2.3.}
As we already noted above, all option-based models with non-optimal-based borrower’s liability in the literature are effectively discrete-time models. Therefore our model is the first fully continuous-time model to use an option-based specification of prepayment. The other discrete-time option-based and MRB mortgage models can be viewed as various numerical approximations of versions of our continuous model, thus making it possible to speak about the “effectiveness” of the methods with respect to their continuous counterparts. As an example we consider Stanton’s option-based model [43] in section 5.1 and find its continuous-time limit. As will be seen, the intuitive econometric numerical procedure employed by Stanton in [43] (and by Dawning, Stanton and Wallace in [11], which is a straightforward generalization of the procedure employed by Stanton in [43]) is a variant of the fractional step numerical method\(^8\) of the first order of accuracy in time applied to our semi-linear “mortgage” PDE.

In our general framework mortgage models differ due to the specification of the prepayment intensity \(\gamma_t\). The most important factor driving prepayment is the refinancing incentive of the borrower, for which we use notation \(\Pi_t\). This quantity models how profitable for the borrower refinancing at time \(t\) is (or the borrower’s perception of the profitability). The likelihood of prepayment will be driven by it in the first place. Therefore we assume that the prepayment intensity \(\gamma_t\) is a function of \(\Pi_t\), i.e., \(\gamma_t = \gamma_t(\Pi_t)\). It can depend on other factors, but it is dependence on \(\Pi_t\) that plays the crucial role, and in this paper we concentrate only on this dependence. Because of the dominance of the factor \(\Pi_t\) in determining the prepayment rates, we base our classification of mortgage modeling approaches on the way \(\Pi_t\) is constructed. Our study of the refinancing incentive \(\Pi_t\) produces new approaches and gives interesting results with consequences for calibration and specification of the prepayment intensity \(\gamma_t\). In particular, we show that the option-based approach, in the way it is applied in the current literature (we call it “traditional”), has a “flaw.” As an alternative, we propose a “reduced” option-based approach, which promises better data fit, has realistic implied transaction costs, and does not require one to evaluate the borrower’s liability as a part of the valuation and calibration procedure.

There is another reason why the dependence of \(\gamma_t(\cdot)\) on \(\Pi_t\) is important. In the current literature on mortgage valuation the function \(\gamma_t(\cdot)\) does not get a lot of attention and is usually assumed to be a step-function with a jump at the boundary which separates profitable/non-profitable states. We show that the implied high transaction costs in the option-based literature are partially due to this jump-specification. Our closer investigation of the prepayment process indicates that the prepayment should be modeled as a multi-staged process (i.e., the prepayment decision-making consists of several stages). We find that this specification can provide a mathematical explanation of the burnout effect. At the same time it implies that the prepayment intensity is not a constant but depends on the history in the burnout-like manner (in particular, it depends on how long the borrower is in in-the-money — the longer, the higher the intensity). It explains why jump-specification implies a very long waiting period for borrowers (Stanton [43] found that borrowers wait an average of almost two years before refinancing).

Our construction of the prepayment function and refinancing incentive explains the “huge” transaction costs implied in the “traditional” option-based literature. Moreover, this construction gives a new idea on modeling mortgage pool heterogeneity. Based on this idea we propose a model to price collateral mortgage obligations (CMO’s) which reduces the complexity of cash flows, making it possible to use a fast PDE approach for valuation.

At the end of section 5.1 we will show how a mortgage model with a one factor interest rate process (because of computational complexity, it is a favorite researcher’s choice for models which consider defaultable mortgages) can be “painlessly” expanded to a model with a simple two factor interest rate process. For example, this approach can reduce the 3-factor discrete model by Kariya, Pliska and Ushiyama [27] to effectively a 2-factor model which can be solved

\(^8\)Or splitting-up method; see, e.g., Marchuk [34] for a description of the method and some applications, and Descombes and Schatzman [10] for a high order convergence scheme for a reaction-diffusion equation based on application of the splitting method.
faster with a PDE approach rather than the Monte-Carlo method that they used.

The model in this paper allows one to also accommodate default of the borrower. We make several comments about this. When working with mortgage default, an assumption that the default hazard function is absolutely continuous (which is implicit with the intensity-based approach) may not be adequate if one works with payments at discrete times. But in this case the researcher can work directly with the hazard default function, which will have jumps at payment days. In other words, the probability of default will be concentrated at discrete points in time. We do not develop the topic here. But, in general, the topic is important because default consideration allows the model to describe observed termination behavior better (compare, e.g., [43] vs. [11] and [41] vs. [42]). One of the ways to elaborate on prepayment/default structure is to consider the termination time as the minimum of prepayment and default random times under an assumption of conditional independence. The notion of conditional independence and the mathematical apparatus for working with the minimum of random times can be found in Bielecki and Rutkowski [2].

In summary, our new formulation of the general mortgage model allows one to see a mortgage from a fresh point of view. It can contribute to a deeper understanding of underlying processes and help researchers to find and apply related known results in default valuation and numerical analysis to mortgage modelling.

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9 Whereas the same assumption is quite reasonable for the prepayment hazard process even in this discrete-payment setting. Indeed, payment days are “special” for the default event: a borrower, if he/she has decided to default, can enjoy his/her underlying possession until the next payment day for free. Therefore the payment days are preferable for default. That does not pertain to the prepayment event. One still can assume continuity of prepayment hazard functions.

10 In the literature on models which acknowledge the presence of (non-optimal) prepayment for both exogenous and endogenous reasons, the prepayment function is represented as the summation of two processes which represent the corresponding parts of prepayment. The conditional independence condition should be stated for this popular splitting of the prepayment process too, but it is never mentioned in the literature on mortgage modeling.
Chapter 2

The Model

2.1 Introduction to Mortgage Securities

In this section we postulate nontechnical assumptions and notations for our basic security — “a mortgage”. Then we briefly review the relationships between a basic mortgage, a pool of mortgages, and mortgage pass-through securities, considering why it can be necessary to price a “basic mortgage” in order to price more complicated mortgage instruments in the case of endogenously modeled prepayment incentive measures.

For descriptions and details on various mortgage securities the reader can consult Fabozzi [15].

2.1.1 A Basic Mortgage

We consider the following contract. A borrower takes a loan of $P_0$ dollars at some initial time and assumes the obligation to pay scheduled coupons at rate $c_t \geq 0$ continuously for duration $T$ of the contract. The loan is secured by the collateral of some specified real estate property, which obliges the borrower to make the payments. For a mortgage originated at time $s$, interest on the principal is compounded according to a contract rate $m_s^{\theta}$, $t \geq s$, where time dependence as the subscript reflects a possible change of the rate as specified by the contract and the superscript $s$ shows the time when this mortgage rate schedule was contracted. When $s = 0$ we omit the superscript, so $m_t = m_0^{\theta}$. When we are dealing with fixed-rate mortgages (no dependence on the subscript) initiated at time $t$ we simply write $m^t$ (the maturity of the mortgage is implicit).

A straightforward argument shows that, in the absence of prepayment, the outstanding principal $P(t)$ at time $t$ equals the future (time-$t$) value of the original principal $P_0$ less the future (time-$t$) value of the cumulative coupon payments, that is,

$$P(t) = P_0 e^{\int_0^t m_\theta d\theta} - \int_0^t c_s e^{\int_s^t m_\theta d\theta} ds.$$  \hfill (2.1)

The borrower has the right to settle his/her obligation during an interval specified by contract\(^1\) and prepay the outstanding principal $P(t)$ in a lump sum. If the borrower prepaays then he is forced to pay concomitant transaction costs. The transaction cost process $F_t$ is assumed to be a defined part of the mortgage model.

**Note on $P(t)$ and $P_t$ notation.** The transaction cost $F_t$ is paid only upon prepayment, i.e., together with the outstanding principal $P(t)$. Therefore some theoretical results will

\(^1\)Commercial mortgages, for example, often have a prepayment lockout period.
be used for “\(P(t) + F_t\)” or “\(P(t)\) only,” depending on what we need to consider: the borrower’s payments or the cash flow received by the investor. To avoid introducing extra notation we will formulate these result for a “general process \(P_t\).” The similarity is intended to remind the reader about the connection between \(P_t\) and the outstanding principal \(P(t)\), but no special assumption like (2.1) is assumed for \(P_t\).

The coupon payment rate \(c_t\) and mortgage rate \(m_t\) can depend on time and the current state of the economy. They can be deterministic functions and even stochastic processes. But they are not independent. In the case of a fully amortized mortgage, i.e., \(P(T) = 0\), equation (2.1) implies that rates \(c_t\) and \(m_t\) must satisfy

\[
P_0 = \int_0^T c_s e^{-\int_0^s m_{s+} d\theta} \, ds.
\]

Although some commercial mortgages may have a balloon payment at the maturity, throughout the thesis we assume that the mortgage is fully amortized. The extension is trivial and can be taken care of within the framework we present.

For example, in the case of the most popular level-payment, fixed rate mortgage, both \(c_t\) and \(m_t\) are constants. If one fixes either rate, \(c\) or \(m\), then the other rate can be found from this equation. Other examples of fixed-rate mortgages are graduated payment mortgage (GPM) and growing equity mortgage (GEM). For a GPM mortgage one chooses a mortgage rate \(m\) and then finds a (deterministic) function \(c_t\) which has some ability to reflect the borrower’s “preferences” when payments should increase and how they should increase. The payment schedule \(c_t\) nevertheless must satisfy (2.2) for a fixed maturity \(T\). GEM mortgages with a fixed mortgage rate \(m\) start with the same payment as a level-payment mortgage (thus avoiding negative amortization which is common for a GPM mortgage), then payments are gradually increased with time. In this case \(c_t\) is an increasing positive function \((c_0 = c)\) and equation (2.2) produces some new maturity time \(T'\) that will be less than the maturity time of the corresponding level-payment mortgage.

An adjustable-rate mortgage is an example of stochastic coupon payment rate \(c_t\) and mortgage rate \(m_t\). The contract mortgage rate \(m_t\) is reset periodically in accordance with some appropriately chosen reference rate; the coupon rate \(c_t\) may be found the same way as it is done in different kinds of fixed-rate mortgages. The reference rate depends on the current state of economy and, therefore, is stochastic.

Another situation where we may want to consider a stochastic coupon rate \(c_t\) is to facilitate curtailment, i.e., partial prepayment. In this case the maturity time \(T\) depends on the whole history of the process \(c_t\) via (2.1). Full prepayment and curtailment have different impacts on mortgage securities. If full prepayment reduces coupon payments in a pool, but leaves the maturity time the same, then curtailment, on the contrary, shortens the maturity time while leaving coupon payments the same. Curtailment has not received the attention it deserves (we can mention Chinloy [5] on the topic). As reported in Hayre and Rajan [21], it amounts roughly to the same portion of total prepayment in a pool as default (though both are still relatively low).

As can be seen, the outstanding principal \(P(t)\) depends on the choice of the contracted mortgage rate \(m_t\) and the contracted coupon payments \(c_t\). In what follows this dependence will be suppressed for notational transparency.

### 2.1.2 Related Contracts

The mortgage market has undergone significant structural changes since the 1980’s. Innovations have occurred in terms of the design of new mortgage instruments and the development of
products that use pools of mortgages as collateral for the issuance of a security. Such securities, called mortgage-backed securities (MBS), remove prepayment uncertainty to some extent but introduce a strong history dependence in the form of so-called burnout.\(^2\) One simple (not model-based) way to model the burnout effect is purely empirical. For example, Schwartz and Torous [41] assume that a pool consists of borrowers with homogeneous characteristics and therefore are forced to take burnout (recall that burnout is a product of a pool’s \textit{heterogeneity}) into account as the explanatory variable \(\ln[P(t)/P^*(t)]\), where \(P(t)\) represents the dollar amount of the pool outstanding at time \(t\), while \(P^*(t)\) is the pool’s principal that would prevail at \(t\) in the absence of prepayments but reflecting the amortization of the underlying mortgages. The adequacy of this approach is questionable because the burnout effect is “non-Markovian”; it should depend on the whole history path of prepayments. The modelling approach (e.g., Stanton [43]) is based on the observation that one particular mortgage does not have burnout by definition. Therefore, using conditional independence of individual mortgages, one can incorporate burnout into a model by modelling the prepayment behavior for individual mortgages and constructing a pool’s prepayment behavior as a combination of individual prepayments. A practical way, however, is the mix of these two approaches — one can assume existence of several homogeneous groups of borrowers, each group consisting of borrowers with identical characteristics.

\textbf{Note on heterogeneous pool pricing.} For efficient valuation one can use some quadrature to price a heterogeneous pool. For example, if we assume that borrowers in the pool differ due to their personal transaction cost only, then we can price mortgages on some grid of transaction cost (where the “density” of the grid will depend on desired precision) and then use, say, Simpson’s rule to estimate the value of the pool.

In the first approach due to homogeneity the price of a pool is the number of borrowers in the pool multiplied by the price of one mortgage. In the modelling approach the price of a pool is the summation of the prices of all individual mortgages in the pool, while in the mixed approach it is a weighted average of them. In both cases the price of a pool, and therefore the price of a mortgage pass-through security,\(^3\) is based on a “basic mortgage”.

But what about other securities, collateralized mortgage obligations, for example? The cash flow of such securities can be very complicated and numerical calculations can be demanding. In order to “clear up the road” for using the forward-looking (quasi) Monte-Carlo method, which is suitable for handling any possible tangled historical dependence of a security’s cash flow, it is desirable to remove future dependence of prepayment behavior, i.e., to define prepayment behavior as a function of current information (this way McConnell and Singh [36] priced CMO’s). With the reduced MRB approach, i.e., with an exogenously modeled mortgage rate process,\(^4\) there is no such problem. But in the case of endogenously modeled prepayment one still needs to work with a “basic mortgage” to do that. Namely, in the option-based case one should price individual liabilities to borrowers (which can be priced as a mortgage with a broad interpretation of the transaction cost) and in the MRB case one should solve the equation “the price of a newly originated generic pool of mortgages (which is equivalent, as we just discussed, to pricing an individual mortgage) is zero” for the mortgage rate process. Therefore the valuation of one mortgage (or a basic mortgage, as we called it in the previous section) is of fundamental importance and, keeping in mind possible generalizations to MBS, in the present paper we will deal only with a model for one mortgage with an emphasis on the way the mortgage prepayment

\(^2\)If the pool experienced a wave of prepayment then it is likely that most of those who used this refinancing opportunity are “fast,” financially astute borrowers, whereas most of those who remain are “slower” borrowers. Therefore, in the presence of another refinancing opportunity, the pool is expected to be less active — the pool is “burned out”.

\(^3\)A holder of this security receives a fixed portion of cash flow from the pool.

\(^4\)For example, the 10 year T-note yield (it is “known today”) is often used as a proxy for the mortgage rate process.
2.2 Mathematical Apparatus

2.2.1 Valuation of Securities with Liquidity Risk

We formalize our setup by introducing a completed filtered probability space \((\Omega, G, \{G_t\}_{t \geq 0}, Q)^5\), where the \(\sigma\)-algebra \(G_t\) represents all observations available to an investor at time \(t\), \(\Omega\) is a set of all possible outcomes and \(Q\) is the probability on \(G\) \(\big( \supseteq \bigcup_{t \geq 0} G_t \big)\). The prepayment time, for which we use notation \(\tau\), is then a positive stopping time on this filtered probability space (i.e., at any arbitrary time \(t\) we can tell if prepayment “occurred” given information \(G_t\)).

We introduce information concerning only the timing of the prepayment as the filtration \(\mathcal{D}_t = \sigma(I_{\tau \leq u}| u \leq t)\). Now given \(\mathcal{D}_t\) and the original filtration \(G_t\), we are interested in decomposing \(G_t\) into \(\mathcal{D}_t\) and an additional filtration \(\mathcal{F}_t\). Formally, it can be defined as a solution of the equality \(G_t = \mathcal{D}_t \vee \mathcal{F}_t\).\(^6\) We accept this complementary filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) as given. In the financial interpretation, \(\{\mathcal{F}_t\}_{t \geq 0}\) is assumed to model the flow of observations available to the lender prior to the prepayment time \(\tau\). Given only this information \(\{\mathcal{F}_t\}_{t \geq 0}\), the lender cannot anticipate the prepayment since he/she does not have complete information about the borrower (such as intention to move, to divorce, etc.) and/or the borrower does not prepay as soon as it is profitable to do so (see, e.g., Hayre and Rajan \[21\]). Thus, we can assume that prepayment time \(\tau\) is not an \(\mathcal{F}_\tau\)-stopping time. This situation is the same as for default time in reduced-form modeling of credit derivatives. The technique is standard and we refer the reader to Jeanblanc and Rutkowski \[25\] for detailed treatment of the topic.

Throughout this paper we assume that all given processes are positive and \(\mathcal{F}_t\)-progressively measurable. The latter condition is purely technical and is needed to have \(\mathcal{F}_t\)-adapted time-integrals of the processes. One simple (and sufficient for most applications) condition for a process to be \(\mathcal{F}_t\)-progressively measurable is that this process is adapted and right- or left-continuous (see, e.g., Métivier \[38\]).

A process \(r_t\) will represent a short-term interest rate, so that at any time \(t\) it is possible to invest one unit in a default-free deposit account and “roll-over” the proceeds until a later time \(s\) for a market value at that time of \(e^{\int_t^s r_u du}\).

The Fundamental Theorem of Arbitrage Pricing (see Harrison and Pliska \[20\]) states that absence of arbitrage implies that all securities are priced in terms of this short-rate process \(r_t\) and an equivalent martingale measure. Therefore, it is convenient to assume that the probability \(Q\) in the introduced probability space is this martingale measure and, thus, all expectations in the paper will be taken under this martingale measure \(Q\) without reminder. In a rough intuitive way, \(Q\) can be understood as the market’s insight about probabilities of possible outcomes. The connection between real-world and martingale probability measures will be briefly discussed at the end of this section.

Since our model is free of arbitrage opportunities, it is appropriate to assume (see, e.g., Bielecki and Rutkowski \[2\]) that the time-\(t\) price of a security \(M_t\) that pays a coupon payment continuously with the rate of \(c_I\) up to time \(\tau \wedge T\) and \(P_{\tau}^T\) in a lump sum at time \(\tau\), if \(\tau \leq T\) (e.g., as with a mortgage), equals the expected discounted value of the future cash flow, with expectation with respect to the martingale measure \(Q\). In other words, (formally) we have

\[\mathbb{E}_Q[M_t] = e^{\int_t^T r_u du}P_{\tau}^T.\]
It is easy to see from (2.1) that if coupon and contract mortgage rates \( c_t \) and \( m_t \) are positive \( \mathcal{F}_t \)-progressively measurable processes then the outstanding principal \( P(t) \) enjoys the same properties. The processes \( c_t \) is assumed to be uniformly integrable. From (2.1) and (2.2) follows that \( P(t) \) is uniformly integrable too in this case. Under these assumptions expectation (2.3) is well defined and finite. But (2.3) is not a convenient formula for calculation because of its explicit dependence on the stopping time \( \tau \). The purpose of the rest of this section is to remove this involvement.

The following definition is standard (see Jeanblanc and Rutkowski [25]).

**Definition 1.** The process \( \Gamma_t = -\ln(1 - Q(\tau \leq t|\mathcal{F}_t)) \) is called the hazard process of the random time \( \tau \). Equivalently \( Q(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t} \).

We assume that \( \Gamma_t \) is an increasing process. Because there is no particular convenient or special day for prepayment, \( \Gamma_t \) is assumed to be continuous. Indeed, we make the slightly more restrictive assumption that the process \( \Gamma_t \) is an absolutely continuous process, i.e., \( \Gamma_t = \int_0^t \gamma_t d\theta \) for some process \( \gamma_t \), called the intensity of the random time \( \tau \). In this case definition 1 has certain similarities with the definition of the intensity of a Poisson process, thereby giving us intuition behind the name “intensity” of \( \gamma_t \).

**Note on the connection to illiquid securities.** The expectation (2.3) has been extensively studied in the literature on defaultable securities (see, for example, Bielecki and Rutkowski [2], Jeanblanc and Rutkowski [24] and [25] on the subject). The timing of default \( \tau \) is uncertain given the information available to investors. But, nevertheless, the likelihood of default is influenced by this information. This is reflected by the \( \mathcal{F}_t \)-adapted intensity \( \gamma_t \). Our idea is that a similar set-up can be applied to securities with “liquidity risk.” Here under “liquidity risk” we understand the risk not to be able to execute transaction when one plans to do it, i.e., not the way it is understood in the academic literature (as ask and bid spread) but the way practitioners call it. For example, if one wants to execute an illiquid option (i.e., \( \tilde{I} \)), the investor may not be able to do it at the right time or may miss the opportunity altogether. The mortgage prepayment option is an example of a very illiquid security for the mortgagor.

Now we are in position to use the following result which is borrowed from the credit-risk literature (e.g., we can reformulate proposition 8.2.1 from Bielecki and Rutkowski [2]). It removes explicit involvement of \( \tau \) by employing its intensity process \( \gamma_t \).

**Theorem 1.** The value of the security \( M_t \) admits the following representation

\[
M_t = \mathbb{E} \left[ \int_t^{\tau \wedge T} c_s e^{-\int_s^\tau r_u du} ds + \mathbb{I}_{\{t < \tau \leq T\}} P_\tau e^{-\int_\tau^T r_u du} \right | \mathcal{F}_t \].
\]

(2.3)

**Note on discrete coupon payments.** If one assumes discrete coupon payments, then the

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\[\text{Footnote 4:} \] For example, for defaultable mortgages with discrete payments it could be reasonable to assume that default times are distributed at due dates, in which case the due dates would be “special” for default. We work with continuous coupon cash flows.

\[\text{Footnote 5:} \] If \( \tau \) is the first jump of a Poisson process with an intensity function \( \gamma(t) \) then \( Q(\tau > t) = e^{-\int_0^t \gamma(\theta) d\theta} \). Etymology of the term “intensity” (or “hazard rate”) for \( \gamma(t) \) came from the fact that for small \( \Delta t \) we have \( Q(t < \tau \leq t + \Delta t | \tau > t) = \gamma(t) \Delta t + o(\Delta t) \) for almost all \( t \in \mathbb{R}_+ \).
analogous result can follow from the same proposition 8.2.1 in Bielecki and Rutkowski [2]) with a dividend process specified as a step-function. This would be the framework, for example, with (non-optimal prepayment version of) the set-up of Kau, Keenan, Muller and Epperson [32]. It can be analyzed along the same lines in what follows.

**Corollary 1.** Let \( P(t) \) is an outstanding principal satisfying (2.1) and (2.2). Then the following special version of equation (2.4) with \( P_1 := P(t) \) is valid for \( \tau > t \):

\[
M_t = P(t) + \mathbb{E} \left[ \int_t^T (m_s - r_s) P(s) e^{-\int_t^s (r_u + \gamma_u) d\theta} ds \mid \mathcal{F}_t \right].
\]

**Proof.** Assume first that \( c_t \) and \( m_t \) are continuous and therefore, from (2.1), we get that \( P(t) \) is continuously differentiable with respect to \( t \). We integrate the second term in the expectation of (2.4) by parts as follows:

\[
\int_t^T P(s) \gamma_s e^{-\int_t^s (r_u + \gamma_u) d\theta} ds = -\int_t^T P(s) e^{-\int_t^s (r_u + \gamma_u) d\theta} \frac{d}{ds} e^{-\int_t^s \gamma_u d\theta} ds,
\]

where we used the amortization condition \( P(T) = 0 \). From (2.1) we get \( P'(t) = m_t P(t) - c_t \). Combining and substituting the above results back into (2.4) we prove the corollary for continuous processes \( c_t \) and \( m_t \).

Now assume that \( c_t \) and \( m_t \) are arbitrary processes. Then there exist sequences of continuous processes \( c^n_t \) and \( m^n_t \) so that they converge to \( c_t \) and \( m_t \) in \( L^1(\Omega) \) a.s. The processes \( c^n_t \) and \( m^n_t \) can be chosen in such a way that they satisfy equation (2.1) and are bounded by \( c_t \) and \( m_t \). The corollary is already proven for continuous processes and, thus, we have the equality of the expectations in (2.4) and (2.5). Now all that we have to do to prove the corollary for the original \( c_t \) and \( m_t \) is to take limit of both sides of that equality for \( n \to \infty \) and change the limits and the expectations. The last manipulation is justified by the dominated convergence theorem. \( \square \)

**Note on applications of Theorem 1 and Corollary 1.** If \( P_t \) is the outstanding principal \( P(t) \) and \( \tau \) is called the prepayment time, then the security in Theorem 1 should be called a mortgage. But Theorem 1 is much more general. For example:

- Later in section 2.2.2 we will use Theorem 1 to define the mortgagor’s liability.

- Instead of representing the whole coupon payment, \( c_t \) can be used for a specified part of the coupon payment, as in pricing interest only (IO) and principal only (PO) strips for example. Specifically, the interest only part, i.e., the part of the payment which covers interest on the principal \( P(t) \), is \( m_t P(t) \). The rest of the coupon payment \( c_t - m_t P(t) \) goes to pay off the principal itself. Therefore the values of an IO strip, for instance, is given by formula 2.4 with substitution of its “\( c_t \)” and “\( P(t) \)” processes with appropriate payments, namely, with \( m_t P(t) \) and 0 respectively. Hence the value of an IO strip is given by:

\[
IO(t) = \mathbb{E}_{(\tau > t)} \left[ \int_t^T m_s P(s) e^{-\int_t^s (r_u + \gamma_u) d\theta} ds \mid \mathcal{F}_t \right].
\]
• It can be used to price complicated mortgage-backed securities such as CMO’s. For example, say we receive a cash flow from a pool of mortgages only up to the \( n \)th prepayment in the pool. Then, reformulating (2.5), which is another representation of (2.4) stated in the corollary 1, the price of the security is given by

\[
M_t = P(t) + \mathbb{E} \left[ \int_t^T (m_s - r_s)P(s)Q(\tau^n > s \mid \mathcal{F}_s)e^{-\int_t^s r_s d\theta} ds \mid \mathcal{F}_t \right],
\]

(2.6)

where \( \tau^n \) is the time of the \( n \)th prepayment and \( Q(\tau^n > t \mid \mathcal{F}_t) \) can be calculated via probabilities of individual borrowers (see section 5.3 for further details).

Note on implicit dependence. The above result does not mean that the price of a mortgage does not depend on a distribution of scheduled payments \( c_t \), because the coupon rate implicitly enters the formula via (2.1) through outstanding principal \( P(t) \) (the same way the mortgage rate \( m_t \) implicitly enters formula (2.4)). For example, the price of a level-payment fixed-rate mortgage generally will not be the same as the price of other types of mortgages, such as graduated payment, growing equity or tiered-payment mortgages, even though they share the same fixed contracted mortgage rate \( m_t \).

The corollary shows that a mortgage can be viewed as a defaultable interest-rate swap on notional amount \( P(t) \). The parties exchange interest payments calculated according to the scheduled (usually fixed) rate \( m_t \) and interest rate \( r_t \) up to possible “default” (i.e., prepayment in our case), the timing of which is driven by the intensity \( \gamma_t \).

Note on the real world and martingale measures. In the literature on mortgage valuation, authors often capture the stochastic nature of prepayments with an empirically estimated prepayment function of some state process such as the borrower’s prepayment incentive, loan-to-value ratio, interest rates, etc. The common feature of this literature is that the prepayment function is assumed to be the same under the real world and martingale measures. Jarrow, Lando and Yu, in their work \cite{23} on default risk, argue that this equivalence is an example of an implicitly applied assumption of conditional diversification. Briefly (and rephrasing the authors so our wording is in terms of mortgage prepayment rather than default), the notion of conditional diversifiability requires that conditioning on the evolution of the state processes, the prepayment processes of borrowers are independent of each other. This captures the idea that once the systematic parts of prepayment risk have been isolated, the residual parts represent idiosyncratic, or borrower-specific, shocks that are uncorrelated across borrowers. Examples of such shocks include divorce, acquisition of a new job or its loss, advent of a new family member, etc. As the nonsystematic risk is not priced, it justifies the practice of using an empirically estimated prepayment function for valuation purposes.

For ourselves we can add that the creation of mortgage-backed securities, namely, the practice of pooling individual mortgages, can be seen as the “physical” implementation of the above argument. As such, the prepayment intensity \( \gamma_t \) can be viewed as a model for prepayment. In general, we can state “mortgage model” = “intensity model” = “prepayment model.” Of course, questions about whether the prepayment intensity is invariant under a change of the probability measure deserves deep empirical research.

2.2.2 Definition of Prepayment Intensity for Option-Based Approaches

In the option-based approaches the prepayment intensity explicitly depends on a “security” which is priced with the help of formula (2.4) (it may be the mortgage price or the borrower’s liability — both are defined with the help of Theorem 1). Therefore (2.4) is not, actually, a formula, but
an equation. In this section we prove existence and uniqueness of a solution to this equation in the case when the prepayment intensity \( \gamma_t \) is a function of the \( t \)-time price of the underlying security \( M_t \), i.e., \( \gamma_t = \gamma_t(M_t) \).

This is an important step in option-based modeling since after one solves equation (2.4) the intensity is defined as an \( \mathcal{F}_t \)-adapted process, i.e., the time \( t \) prepayment behavior is defined as a functional of the information available at that time. Therefore the road is open to use a straightforward (quasi) Monte-Carlo simulation to price various types of mortgage-backed securities (not only mortgage pass through securities). In particular, we can handle collateralized mortgage obligations, which can have extremely complicated cash flows.

Since the increase of the borrower’s liability (the mortgage price) signals that the loan the borrower has gets more expensive, it is reasonable to assume that the likelihood of prepayment grows as \( M \) increases having all other factors the same. Therefore, throughout the thesis, if the “sensitivity” of the prepayment intensity to profitability of prepayment is expressed in terms of \( M_t \) as above then \( \gamma_t(M) \) is assumed to be increasing in \( M \) (this will be postulated in term of a “refinancing incentive” in the first paragraph of chapter 3). Additionally to the current assumptions we assume that \( \gamma_t(x) \) is bounded and uniformly Lipschitz continuous.

**Lemma 1.** Let the processes \( A_t(x) \) and \( B_t(x) \) depend on the parameter \( x \). Next, assume for all \( t \in [0,T] \)

\[
|A_t(x) - A_t(y)| \leq N_t|x - y|, \quad |B_t(x) - B_t(y)| \leq L_1|x - y|,
\]

where \( N_t \) is a uniformly integrable process, in particular, \( \mathbb{E}[N_t] \leq L_2 \), and \( L_1 \) and \( L_2 \) are constants. Suppose \( A_t(x) \) satisfies the integrability condition \( \mathbb{E} \left[ \int_t^T A_s(x_s) \, ds \right] \leq L_3 \) for any bounded \( \mathcal{F}_t \)-measurable process \( x_t \) and some finite constant \( L_3 \). Then there exists a solution to equation

\[
x_t = \mathbf{J}[x]_t := \mathbb{E} \left[ \int_t^T A_s(x_s)e^{-\int_t^s B_s(x_s) \, ds} \, ds \middle| \mathcal{F}_t \right]. \tag{2.7}
\]

This solution is unique in the set of bounded \( \mathcal{F}_t \)-adapted processes.

**Proof.** Let \( D_\alpha^b \) be a space of a bounded \( \mathcal{F}_t \)-adapted processes with the norm

\[
||x_t||_\alpha = \sup_{t \in [t,T], \omega \in \Omega} e^{-\alpha(T-t)}|x_t(\omega)|.
\]

The theorem is equivalent to the problem of existence and uniqueness of the fixed point \( x_t \in D_\alpha^b \) of the operator \( \mathbf{J}[x] \), which is defined as the right-hand side of the equation (2.7). Clearly, \( \mathbf{J} \) maps \( D_\alpha^b \) into itself. Moreover, the operator is a contraction in that space for \( \alpha \) large enough. Indeed, for bounded \( \mathcal{F}_t \)-adapted processes \( x_t \) and \( y_t \) we have

\[
|\mathbf{J}[x]_t - \mathbf{J}[y]_t| \leq \mathbb{E} \left[ \int_t^T |A_s(x_s) - A_s(y_s)| e^{-\int_t^s B_s(x_s) \, ds} \, ds \middle| \mathcal{F}_t \right] + \mathbb{E} \left[ \int_t^T A_s(y_s) \left| e^{-\int_t^s B_s(x_s) \, ds} - e^{-\int_t^s B_s(y_s) \, ds} \right| \, ds \middle| \mathcal{F}_t \right]
\]

\[
\leq \mathbb{E} \left[ \int_t^T \left( N||x_t - y_t||_\alpha e^{\alpha(T-s)} + A_s(y_s) \int_t^s L_1||x_\theta - y_\theta||_\alpha e^{-\alpha(T-\theta)} \, d\theta \right) d\theta \right]
\]

\[\overset{10}{10}\text{Lipschitz continuity of } \gamma_t(\cdot) \text{ is a quite natural assumption; we discuss this topic later in section 4.1.1.}\]
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\[ \frac{L_2 + L_1 L_3}{\alpha} e^{\alpha (T-t)} \| x_t - y_t \|_\alpha. \]

Taking the norm \( \| \cdot \|_\alpha \) of this inequality we see that operator \( J \) is a contraction if \( \alpha > L_2 + L_1 L_3 \):

\[ \| J[x]_t - J[y]_t \|_\alpha \leq \beta \| x_t - y_t \|_\alpha, \]

where the constant \( \beta = (L_2 + L_1 L_3)/\alpha < 1 \). We thus conclude that the equation \( x_t = J[x]_t \) has a unique solution in the set of bounded \( \mathcal{F}_t \)-adapted processes.

\section*{Lemma 1}

\textit{Proof.} The result is a straightforward consequence of Lemma 1. The conditions of Lemma 1 are easily verified. \qed

\subsection{2.2.3 Diffusion State Process}

To formulate the mortgage model in a continuous-time diffusion state process setting, which is popular in financial applications, we assume that all relevant factors in the model (i.e., \( r_t, e_t, P_t \) and \( \gamma_t \)) are deterministic functions of time \( t \) and a state process \( X_t = (X_t^1, ..., X^n_t) \), for some \( n \). Additionally, if the option-based approach is under consideration, \( \gamma_t \) may also depend on the underlying security \( M_t \) (it may be the borrower’s liability \( L_t \) which is defined in section 3.1.2 or the price of the mortgage \( M_t \) itself — both are defined with the help of (2.4)). To avoid long expressions we often will use notation \( f_t \) for \( f(t, X_t) \). So that instead of writing \( r(t, X_t) \) and \( \gamma(t, X_t, M(t, X_t)) \) we use notations \( r_t \) and \( \gamma_t(M_t) \).

The state process \( X_t \) is a diffusion process following the stochastic differential equation

\[ dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_s = x \in D \]

with an \( m \)-dimensional Brownian motion \( W_t \in \mathbb{R}^m \) and functions \( \mu : [0, T] \times D \to \mathbb{R}^n, \sigma : [0, T] \times D \to \mathbb{R}^n \times \mathbb{R}^m \), where \( D \) is a domain (i.e., open and connected set) in \( \mathbb{R}^n \). If the dependence of \( X_t \) on the initial conditions \( (s, x) \) will be important, we will use notation \( X^{s,x}_t \) will be used, i.e., in particular we have \( X^{s,x}_T = x \). If the initial time (and the initial position) will be fixed and clear from the context we skip the dependence on it and use notation \( X^x_T \) (notation \( X_t \)).

We assume the following conditions on the coefficient of the stochastic differential equation:

\textbf{(A)} The functions \( \mu(t, x) \) and \( \sigma(t, x) \) are locally Lipschitz continuous in \( x \in D \), uniformly in \( t \in [0, T] \).

\textbf{(B)} The solution \( X_t \) of neither explodes nor leaves \( D \) before \( T \), i.e., \( Q[\sup_{t \in [s, T]} |X_t^{s,x}| < \infty \) and \( X^{s,x}_t \in D \) for all \( t \in [s, T] \) and that this solution is continuous as a function of \( (t, s, x) \) (see Theorem II.5.2 of Kunita [33]).

These assumptions guarantees that the strong solution \( X^{s,x}_t \) of SDE (2.8) exists on the interval \([s, T]\) and that this solution is continuous as a function of \((t, s, x)\) (see Theorem II.5.2 of Kunita [33]).

Finally, to state equivalence of the martingale and PDE approaches we need additionally these assumptions:

\textbf{(C)} The functions \( r, c, P \) and \( \gamma \) are Hölder continuous in \((t, x) \in [0, T] \times D \); the function \( \gamma \) is uniformly Hölder continuous in \( M \) for all \((t, x) \in [0, T] \times D \).

\textbf{(D)} \( \sigma(t, x) \neq 0 \) for all \((t, x) \in [0, T] \times D \).\footnote{Recall that \( \gamma_t(M) \) is assumed to be increasing in \( M \). See section 2.2.2.}
CHAPTER 2. THE MODEL

Let operator \( A \) be the generator of the diffusion state process \( X_t \), i.e.,

\[
A := \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i}.
\]

Then the Feynman-Kac representation states that to evaluate the underlying security \( M_t \), i.e., to solve equation (2.4), which in the present setting is written as

\[
M_t(x) = \mathbb{E} \left[ \int_t^T (c_s + P_s \gamma_s(M_s)) e^{-\int_s^t [r_s + \gamma_s(M_s)] ds} d\theta \right| X_t = x],
\]

one can solve the following backward semi-linear parabolic partial differential equation

\[
\frac{\partial M}{\partial t} + AM - [r + \gamma(M)]M + c + P\gamma(M) = 0, \quad 0 < t < T
\]

\[
M_T(x) = 0, \quad x \in D,
\]

where we left out the dependence of \( M, r, c, \gamma, P \) on time \( t \) and the value of the state variables \( x \) for notational simplicity and to underline the nonlinear dependence of \( \gamma \) on \( M \) (we will keep this practice in what follows). Under the stated assumptions a classical solution \( M_t \) (i.e., \( M_t \in C^{1,2}([0, T] \times D) \)) to equation (2.10) exists and is unique. The proof of this result can be deduced from Theorem 1 of Heath and Schweizer [22] which is proved for a linear case, but easily generalized to our semi-linear set-up with the help of the fact that condition (A3) there, modified to a semi-linear case, can be verified, say, by Theorem 7.4.9 of Friedman [18].

**Note on the technical conditions.** There are many sets of sufficient conditions for equivalence of martingale and PDE approaches (Feynman-Kac type results). The most popular conditions (see, e.g., appendix E of Duffie [12] for an overview of the conditions and references) are quite restrictive and are not satisfied for some popular set-ups. For example, in the Black-Scholes model \( \sigma(t, x) \) is not uniformly elliptic since \( \sigma(t, x) := \sigma x \) for \( x = 0 \); in the CIR interest rate model \( \sigma(t, x) \), additionally, is not uniformly Lipschitz continuous; and in the Vasicek interest rate model \( \sigma(t, x) := \sigma \sqrt{x} \) does not satisfy even a linear growth condition.\(^{12}\)

However, the equivalence is almost universally assumed in financial mathematics as well as in the academic mathematics literature. For example, Friedlin [17] defined a generalized solution for linear and quasi-linear (the MRB approaches and the option-based approaches in what follows) parabolic equations and studied their properties with the help of a probabilistic approach.

Our technical assumptions can handle degeneracy (and non-uniformity of a Lipschitz constant) of the coefficients at the boundary of \( D \), and it does not require coefficients of SDE (2.8) to be bounded or satisfy linear-growth condition. Our set-up is general enough to work for many popular and reasonable set-ups.

While Conditions (A), (C) and (D) are easy to verify, Condition (B) should be checked individually based on a careful study of the process \( X_t \). For example, if we take the Vasicek model which allows the interest rate to be negative (although with a small probability for reasonable parameters), then the domain \( D \) is the whole real line \( \mathbb{R} \) and it is easy to see that the process does not blow up. For the CIR one-factor interest rate model

\[
\frac{dr_t}{r_t} = (a(b-r_t))dt + \sigma \sqrt{r_t}dW_t,
\]

the domain \( D = (0, \infty) \); thus the model implies that the interest rate is positive. It is known (this is a special case of results in Feller [16]) that Condition (B) in this case, i.e., that \( r_t \) stays

\(^{12}\)In the case of time-independent coefficients these models admit analytical solutions. That is not generally the case for time-dependent coefficients.
in $D$, is satisfied if $2ab/\sigma^2 \geq 1$. If this condition is not satisfied then we need to impose a boundary condition at $r = 0$. ■

**Note on a stochastic intensity.** We can assume that the prepayment intensity depends on the underlying security $M_t$ through some of the components of $X_t$ (only or along with the straightforward dependence), i.e., some coefficients in (2.8) can depend on $M_t$. The existence of such components can be justified by the presence of randomness in the prepayment rate even if conditioned on “standard” information (interest rates, unemployment, loan-to-value ratio, etc.). The source of the randomness can come from the fact that human perception of the world changes with the influence of a lot of factors such as weather, political and social situation in the country, etc., which cannot be removed by diversification. This assumption leads to a quasi-linear PDE (some of the coefficients in operator $A$ depend on $M_t$) as opposed to a semi-linear PDE (2.10). This demands more attention from the researcher as the problem of existence of a solution to quasi-linear PDE’s is more complex than in the case of linear equations and semi-linear equations. ■

A solution of the mortgage equation (2.9) or, equivalently, (2.10) depends on the time $t$, the initial condition $x$ of the diffusion state process $X_t$ (i.e., $X_0 = x$) and the contract mortgage rate $m$, i.e., we have the map $(t, x, m) \mapsto M(t, x; m)$ which is defined by solving (2.9) or (2.10). Later we will need continuity of $M_t$ with respect to $m$. It is stated in the following corollary.

**Lemma 2.** Let $M(t, x, m)$ be a solution of the equation (2.9). Then the map $(t, x, m) \mapsto M(t, x, m)$ is continuous for $(t, x, m) \in [0, T] \times D \times [0, \infty)$.

**Proof.** First, for a function $u(t, x, m)$ on $[0, T] \times D \times [0, \infty)$ we define an operator $J$ as follows:

$$J[u] := E \left[ \int_0^T \left( c_s + P_s \gamma_s(u_s) \right) e^{-\int_s^t [c_{s+\gamma_s(u_s)}] ds} dX_s \right] = x]$$

where the notation $u_t$ is used for $u(t, X_t, m_t)$. Then the solution $M(t, x, m)$ of the equation 2.9 is a fixed point of the operator $J$, i.e. $u = J[u]$.

The operator $J$ maps $C([0, T] \times D \times [0, \infty))$ into itself. Indeed, from Theorem II.5.2 of Kunita [33] we can conclude that the conditions stated in the present section guarantee not only existence of a solution $X^{s, x}_t$ to the diffusion equation (2.8) for any initial conditions $X^{s, x}_0 = x \in D$, but also continuity of the map $(t, s, x) \mapsto X^{s, x}_t$ on $[s, T] \times [0, T] \times D$. This map is uniformly continuous on any compact subset of $[s, T] \times [0, T] \times D$. Therefore $\int_0^T [r(\theta, X^{s, x}_\theta) + \gamma(\theta, X^{s, x}_\theta, u(t, X^{s, x}_\theta, m))] d\theta$ and $\int_0^T [c_t + \gamma_t(u_t, P_t)] ds$ are continuous. Since the expression under the expectation in (2.9) is uniformly integrable (it follows from uniform integrability of $c_t$ and $P_t$ which we assumed in section 2.2.1), the function $J[u](t, x, m)$ is continuous for $(t, x, m) \in [0, T] \times D \times [0, \infty)$ from the dominated convergence theorem.

Next, the proof of Lemma 1 can be repeated here to show that the operator $J$ is a contraction on the space of continuous functions $C([0, T] \times D \times [0, \infty))$ with the norm

$$||u||_\alpha = \sup_{(t, x, m) \in [0, T] \times D \times [0, T]} e^{-\alpha(T-t)} |u(t, x, m)|$$

for some $\alpha$ large enough. Therefore, the unique fixed point of the operator $J$ (which is guaranteed by Lemma 1) lies in the space of continuous functions $C([0, T] \times D \times [0, \infty))$. ■

2.2. **MATHEMATICAL APPARATUS**
2.3 Mortgage Rate

It is of a vital practical interest for Wall Street to define the mortgage rate endogenously, i.e., to find what mortgage rate is implied by the current (riskless) yield curve and the prepayment behavior of a representative mortgagor. Consider the case of the fixed-rate mortgages, i.e., $m^t$ is a mortgage rate of a fixed-rate mortgage originated at time $t$. Then this can be done through the postulate that at origination the value of a mortgage should be equal to the initial principal. For example, from (2.5), using the arbitrage argument that $M_0 = P_0$, we can get that the mortgage rate $m^t$ of a fixed-rate default-free mortgage at time $t$ is a weighted average of the risk-free interest rate over the life-time $T$ of the contract, that is,\textsuperscript{13}

\[ m^t = \frac{1}{E} \left[ \frac{\int_0^T r_s (s-t)e^{-\int_0^t (r_s + \gamma_{\theta}) d\theta}}{\int_0^T P(s-t)e^{-\int_0^t (r_s + \gamma_{\theta}) d\theta}} ds \bigg| \mathcal{F}_t \right]. \tag{2.11} \]

Unfortunately, this is an equation rather than a formula because at least $P(s)$ and thus the right-hand side of (2.11) depend on the mortgage rate too. The prepayment intensity $\gamma_t$ may also depend on the mortgage rate. This dependence can be quite different for different approaches. The prepayment intensity $\gamma_t$ can be modeled in such a way that it depends only on the contract mortgage rate $m^t$ for all $s \in [t, T + t]$ (this is the case of, say, the traditional or reduced option-based approach where $\gamma_t$ depend on $m^t$ implicitly through the mortgage price; see the next chapter) or it can depend on the future mortgage rates $m^s$ as well (the MRB approaches). The latter case (MRB) greatly complicates the problem because equation (2.11) is a functional equation rather than just a nonlinear equation in one variable as in the former case (the option-based approach).

If $\gamma_t$ depends on the price of the mortgage $M_t$ (the reduced option-based approach in what follows\textsuperscript{14}) then we have the following theorem on the existence of a solution.

**Theorem 3.** Consider the diffusion-state process specification of section 2.2.3 for $\gamma_t = \gamma(t, X_t, M_t)$. Assume that $X_t$ is a stationary process, i.e., $\mu$ and $\sigma$ in (2.8) do not depend on time $t$. Then a continuous solution $m(x)$ of the mortgage rate equation exists with specification,\textsuperscript{15}

\[ m(x) = \frac{1}{E} \left[ \frac{\int_0^T r_s P(s) e^{-\int_0^t (r_s + \gamma_{\theta}(M^s_M)) d\theta} ds \bigg| X_0 = x}{\int_0^T P(s) e^{-\int_0^t (r_s + \gamma_{\theta}(M^s_M)) d\theta} ds \bigg| X_0 = x} \right]. \tag{2.12} \]

**Proof.** From Lemma 2 of section 2.2.3 we have that $M(t, x, m)$ is continuous for $(t, x, m) \in [0,T] \times D \times [0,\infty)$. From Theorem II.5.2 of Kunita [33] we can conclude that the conditions of section 2.2.3 guarantees existence of a solution $X_t^x$ to the diffusion equation (2.8) of section 2.2.3 for any initial conditions $X_0^x = x \in D$. Additionally the theorem states that a realization $X_t^x$ is continuous as a function of $(t, x) \in [0,T] \times D$. Therefore $M(t, x, m)$ and $X_t^x$ are uniformly continuous on any compact subset of $[0,T] \times D \times [0,\infty)$ and we have that $\int_0^T [r(\theta, X_{\theta}^x) + \gamma(\theta, X_{\theta}^x, M_\theta^x)] d\theta$

\textsuperscript{13}Mortgages are mostly traded in pools. Therefore, if one assumes heterogeneity of borrowers, the denominator and the numerator of equation (2.11) should be “averaged” over a possible “risk-neutral” distribution of borrowers’ prepayment processes $\gamma_t$.

\textsuperscript{14}This is equally applicable to the traditional option-based approach.

\textsuperscript{15}We use notation $M^s_M$ for the mortgage price to remind the reader about the ulterior dependence of the mortgage price on the mortgage rate $m$. 
and $\int_0^T r(\theta,X^0_{\theta,x})ds$ are continuous. The expression under expectation in the denominator of (2.11) is bounded by $P_0$ and, thus, is uniformly integrable. It is easy to show that the expression under the expectation in the numerator is uniformly integrable too:

$$\int_0^T r_x P(s) e^{-\int_0^s (r_x + \gamma^0(\hat{M}_t^{m(x)}))d\theta} ds \leq P_0 \int_0^T r_x e^{-\int_0^s r_x d\theta} ds < P_0$$

Therefore the expectations in the denominator and numerator of the mortgage rate equation (2.12) are continuous in $(x,m) \in D \times [0,\infty)$. Now let us fix $x \in D$. The existence of a solution of the mortgage rate equation follows from the fact that the right hand side of (2.11) is uniformly bounded from above and positive for all $m$ for a fixed $x$ and thus the equation (2.11) has a fixed point. The continuity of this solution $m(x)$ follows from shown continuity of the right hand side of (2.11) as a function of $m$ and $x$. $\square$
Chapter 3
Specification of Refinancing Incentive

Specification of the prepayment intensity process $\gamma_t$ is a cornerstone of mortgage pricing. It governs the timing of prepayment, the most specific characteristic of mortgage modeling. Let $\Pi$ be a measure of refinancing profitability, i.e., a scalar quantity which measures refinancing incentive of the borrower. The larger the refinancing incentive the more likely the borrower will refinance his/her mortgage. Therefore it is natural to postulate that prepayment intensity $\gamma_t$ is an increasing function of $\Pi$, i.e., $\gamma_t(\Pi_1) \leq \gamma_t(\Pi_2)$ for $\Pi_1 \leq \Pi_2$. Existing mortgage models mostly assume that $\gamma_t(\Pi)$ is a constant as a function of $\Pi$ up to a point where $\Pi$ is less than transaction costs and a greater constant or growing linear function after that. The form of $\gamma_t$ dependence on $\Pi$ is a (important!) matter of statistical research, but we will not consider it for an approach classification. Another important question is how to measure the refinancing incentive $\Pi$ itself. This quantity will be a key for model classification. Since most considerations will be done for fixed-rate mortgages, from now on we will assume that the contract mortgage rate is fixed at the origination, i.e., $m_t$ is a fixed mortgage rate for a loan originated at time $t$.

For easier references here we give definitions of all approaches we will consider in the present work.

**Definition 2.** We call a mortgage model **mortgage-rate-based** (MRB for short) model if the refinancing incentive $\Pi_t$ is a function of the current $m_t$ and the contract $m^0$ mortgage rates, i.e., the process $\Pi_t$ is of the form $\Pi_t(m^0, m_t)$. We separate MRB models into two:

- **empirical** — The contract mortgage rate $m^0$ is assumed to be given and the current mortgage rate $m_t$ is empirically (exogenously) defined.
- **endogenous** — The contract mortgage rate $m^0$ and the current mortgage rate $m_t$ are defined by the mortgage rate equation (2.11), i.e., the mortgage rate process $m_t$ is determined by the model endogenously.

**Definition 3.** We call a mortgage model an **option-based** model if the refinancing incentive estimation is based on pricing of underlying securities. If we consider the definition of $\Pi_t$ closer we can distinguish the following kinds of option-based approaches:

- **traditional** — the refinancing incentive is modeled as the difference between the borrower’s liability (defined below) and the outstanding principal, i.e., $\Pi_t = L_t - P(t)$.

---

[1] That does not mean that the coupon payments $c_t$ is a constant. It can change according to some schedule. Examples of such mortgages are graduated payment mortgages and growing equity mortgages. We refer the reader to section 2.1.1 or Fabozzi [15].
• **reduced** — the refinancing incentive is modeled as the difference between the mortgage price and the outstanding principal, i.e., $\Pi_t = M_t - P(t)$.

• **complete** — the refinancing incentive $\Pi_t$ is the difference of the liabilities evaluated with the original and current mortgage rate, i.e., $\Pi_t = L_t(m^0) - L_t(m^t)$.

The approaches can be mixed to get new ones. For example:

**Definition 4.** We call a mortgage model an option-based MRB model if the refinancing incentive $\Pi_t$ is the difference of the liabilities with the original and current mortgage rate which are evaluated over the original term of the modeled mortgage and with the initial principal $P(t)$, i.e., $\Pi_t = L^T_t(m^0) - L^T_t(m^t)$.

The above approach can be “reduced.” This, as we will show, removes dependence on the mortgage rate process $m_t$ for $t > 0$ in the time homogeneous setting:

**Definition 5.** We call a mortgage model a reduced option-based MRB model if the refinancing incentive $\Pi_t$ is the difference of the mortgage prices with the original and current mortgage rates which are evaluated over the original term of the modeled mortgage and with the initial principal $P(t)$, i.e., $\Pi_t = M^T_t(m^0) - M^T_t(m^t)$.

Only the empirical MRB and the traditional option-based approaches were considered in the literature. The other approaches are new to the best knowledge of this author. In this chapter we shall discuss these approaches, ideas behind them, and their important features.

**Note on the extension to multi-mortgage market.** It is worth noting here that the possibility of being able to refinance from a 30-year loan into a 15-year loan or an adjustable-rate mortgage means that the mortgagor can consider a complex mix of rates and monthly payments. However, it is important to see that refinancing choice is nevertheless limited. There is no 23.5-year mortgage to refinance 6.5-year seasoned 30-year loans. To refinance the loan with a 15-year mortgage may be too expensive for the mortgagor, and an adjustable-rate mortgage can be too risky (and it might be more expensive because servicing fees are higher), although this type of loan can be attractive to mortgagors who plan to sell their property soon. Some mortgage modeling approaches (the MRB) recognize this fact of limited choice, while others do not (the traditional or reduced option-based approaches). If it is the former case then we assume that only one kind of mortgage is available. For example, a 30-year fixed-rate mortgage can be refinanced only with another 30-year fixed-rate mortgage.

To account for possibilities of refinancing to several types of mortgages we can consider the prepayment intensity function which depends on several “competing” refinancing incentives. For example, if we have that 15- and 30-year mortgages are available in a market then $\gamma_t(\Pi^1_{15}, \Pi^1_{30})$ can depend on incentives to refinance to 15-year mortgages $\Pi^1_{15}$ or 30-year mortgages $\Pi^1_{30}$. The valuation of 15-year mortgages, perhaps, would be the same as in the case of a one-mortgage-market since it is not likely that anyone would refinance 15-year with 30-year mortgages. Therefore, for purpose of 15-year mortgage pricing, the prepayment intensity can be assumed to depend on the possibility of refinancing only to the same 15-year mortgage. In general, we would price mortgages in multi-mortgage-market in turn of growing maturities, i.e., once the shortest maturity mortgage is priced we can proceed to price longer maturity mortgages.

### 3.1 Traditional Approaches

#### 3.1.1 MRB Approaches

A simple way to measure the prepayment incentive is to compare the contract mortgage $m^0$ with an available for refinancing mortgage rate $m^t$. On the web, for example, one can find a lot of
calculators which can say if it is profitable to refinance (and how much one saves) on the base of mortgage rates information. The approach is intuitively clear and assumes that the borrower considers refinancing to another mortgage of the same type. As Hayre and Rajan [21] report, the fixed-rate 30-year mortgage has retained its popularity as the refinancing vehicle of choice for mortgagors with an existing 30-year loan. That gives a rational for the MRB definition of the refinancing incentive.

The refinancing incentive \( \Pi_t \) can be the difference \( m^0_t - m^t_t \) (models by Kariya and Kobayashi [28], Kariya, Pliska and Ushiyama [27] and Schwartz and Torous [41, 42] assume this specification), the rational expression \( m^0_t/m^t_t \) (as was argued in Hayre and Rajan [21] and Richard and Roll [40], it is able to capture refinancing incentive better then the difference), or it can be given by direct computation of how much the borrower will save on coupon payments with refinancing (Deng [7], Deng and Quigley [8], Deng, Quigley and Van Order [9]). Depending on the type (e.g., a 30-year or 15-year mortgage rate) of the mortgage rate \( m^t_t \) considered, the expressions above approximate/evaluate potential savings from refinancing a \( m^0 \)-fixed-rate mortgage with a (30-year or 15-year mortgage rate) \( m^t \)-fixed-rate mortgage.

The mortgage rate can be defined exogenously (empirically). In this (empirical MRB) case a common choice is to model the mortgage rate as the 10-year Treasury note yield plus a constant. As reported in Boudoukh, Richardson, Stanton and Whitelaw [3], the 10-year Treasury note yield has correlation of 0.98 with the mortgage rate. Belbase and Szakallas [1] found that Libor swap rates are better predictors than the Treasury yield (the correlation of 30-year mortgage rate vs. 10-year Libor swap rate and vs. 10-year Treasury yield are .983 and .915 respectively).

MRB approaches allow easy fit to prepayment data as opposed to option-based approaches. The mortgage rate data are readily available, thus the refinancing incentive is available without additional calculations. We can use this data together with prepayment data to calibrate the prepayment intensity function.

### 3.1.2 Traditional Option-Based Approach

The MRB approach is “naive:” the MRB refinancing incentive \( \Pi_t(m^0_t, m^t_t) \) does not depend on, for example, volatility of the interest rates.\(^2\) As is well known from option pricing theory (mortgages have embedded options!), the value of an option is quite sensitive to changes in volatility of an underlying asset. We may expect that \( \Pi_t \) decreases if the volatility increases (keeping the other factors the same) since the probability of the option being (deeper) “in-the-money” is higher. This feature is fully incorporated in the following approach, which is based on the option pricing idea.

Intuitively saying the price of, say, an American option can be found under an assumption that a holder of the option exercises it as soon as doing so is profitable, i.e., the payoff is greater then the price of the option. In the context of mortgage pricing, if a borrower prepays the mortgage, he/she must pay the outstanding principal \( P(t) \) plus (perhaps substantial) transaction costs and, in return, he/she is liberated from his/her obligations to pay coupons, i.e., he/she “gets back” his/her liability. This idea stands behind the option-based definition of the refinancing incentive.

The liability \( L_t \) itself can be defined with the help of Theorem 1. We note that the distinction between the liability to the mortgagor and the price of asset to the investor is that at time of prepayment a holder of the security receives the outstanding principal \( P(t) \), but the mortgagor pays \( P(t) \) plus a transaction cost, which we denote by \( F_t \) (\( F_t \) is assumed to be uniformly integrable). Therefore the liability \( L_t \) has the interpretation of the price of a security paying coupons \( c_t \) continuously up to time \( \tau \lor T \), and \( P(t) + F_t \) upon prepayment of the mortgage. So we just

\(^2\)One can include this (and other) dependence, but it would be an ad-hoc approach.
use the formula (2.4) with \( P(t) + F_t \) instead of \( P_t \):

\[
L_t = E \left[ \int_t^T \left( c_s + [P_s + F_s] \gamma_s \right) e^{-\int_t^s \left( r_x + \gamma_x \right) ds} \, ds \, \mathcal{F}_t \right].
\] (3.1)

The traditional option-based approach implies that \( \gamma_t = \gamma_t(L_t - P(t)) \) and, thus, (3.1) is not a formula but an equation. The existence of a solution was discussed in sections 2.2.2 and 2.2.3.

**Note on the liability specification.** The liability \( L_t \) is not just the discounted value of the cash flow \( c_t \); it takes into account future prepayment opportunities. It is important to note the presence of \( \gamma_t \) and \( F_t \) in the definition of the liability, i.e., the borrower is aware of the possibilities to miss prepayment opportunities in the future and takes into account the future transaction costs associated with the prepayment. This feature is sometimes overseen in the option-based mortgage modeling literature.

After we solve equation (3.1), the prepayment intensity \( \gamma_t \) is an \( \mathcal{F}_t \)-adapted processes, we are in the framework of section 2.2.1, and can find the price of the mortgage or related mortgage security. The existence and uniqueness of the solution to the liability equation in the case when model factors are diffusions is studied in the forward-backward stochastic differential equations literature (see the expository paper Pardoux [39] for the theory and the application of decoupled FBSDE’s). In the section 2.2.3 the equivalence of PDE and martingale approaches and existence of classical solutions are proved for assumptions which are general enough to include the popular diffusion-state set-ups. For a bounded and uniformly Lipschitz continuous \( \gamma_t \), the existence and uniqueness can be proved in a very general setting (see section 2.2.2).

The traditional option-based approach does not require one to model the mortgage rate process. The knowledge of the contract rate \( c_t \) is sufficient. However, things get more complicated when we calibrate the prepayment intensity function. The borrower liability is not available to us without additional calculations and, therefore, the refinancing incentive data are not “free” as they were in the case of the MRB approaches. Therefore the prepayment intensity calibration procedure should be done together with the liability valuation, and this is numerically an expensive procedure.

### 3.1.3 MRB vs. Option-Based Specification of Refinancing Incentive

An option-based measure of refinancing incentive is based on a comparison of outstanding principal \( P(t) \) and the liability of the mortgagor. Whatever kind of a loan the mortgagor chooses to pay off, the price of the loan at the origination should coincide with \( P(t) \). On the other hand, we have to keep in mind that the borrower probably refinances to another mortgage of the same type and, thus, prolongs the time of repayment of the loan (that may be undesirable for the mortgagor). However, from the option-based approach point of view, the borrower is indifferent to the length of repayment time.

**Example.** The following simple example emphasizes this point. Let us consider a 30-year mortgage. Assume that the risk-free interest rate is deterministic and equals the constant \( r_1 \) from 0 to 15 years, then 0 from 15 to 30 years and the constant \( r_2 \) from 30 years on. The prepayment intensity \( \gamma_t \) is arbitrary, but bounded.

From the mortgage rate equation (2.11) we can conclude that the mortgage rate at time 0 is less then \( r_1 \), i.e., \( m^0 < r_1 \) (the particular value of \( m^0 \) depends on the prepayment intensity

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3We could also change the coupon rate in the mortgage pricing formula to account for servicing fee. This is a relatively small amount and is usually neglected.

4Lipschitz continuity of \( \gamma_t \) is a quite natural assumption; we discuss this topic later in section 4.1.
3.2. NEW APPROACHES

which we do not specify here), and \( m^0 \rightarrow +\infty \) monotonically as \( r_1 \rightarrow +\infty \). From the same equation we can see that at time \( t = 15 \) the mortgage rate \( m_{15} \) is monotonically increasing to \( +\infty \) as \( r_2 \rightarrow +\infty \). The liability \( L_{15} \) is higher then \( P(15) \) because the interest rate \( r_t \) is zero for \( t \in (15,30) \). Moreover, \( L_t - P(t) \) can be made larger then any specified transaction cost by choosing \( r_2 \) large enough. This means that at time \( t = 15 \) it is profitable to refinance the mortgage from the option-based point of view for that value of \( r_2 \). However, choosing \( r_2 \) large enough we can have \( m_{15} > m^0 \). Therefore, at time \( t = 15 \) refinancing to another 30-year mortgage is unprofitable from MRB point of view. It is unlikely that the borrower will follow “option-based” advise which causes him to refinance to another mortgage with a higher rate.

This shows that if the choice of refinancing vehicles is limited then the option-based measure of refinancing incentive is not adequate. This measure implies total availability of funds to a borrower and neglects the borrower’s preferences and ability to pay a certain cash flow (a borrower chooses between 30-year and 15-year mortgages according to his/her income and value of the loan). The outstanding principal \( P(t) \) is compared with the liability, which is based on the behavior of economic variables over the period of time \([t, T]\), thus making the traditional option-based refinancing incentive measure “comparable”5 with refinancing the current mortgage into a \((T - t)\)-year fixed-rate mortgage (which is valid over the same period of time), the choice of which is not readily available to a residential mortgagor.

Another reason why the MRB choice of incentive measure may be more appropriate then the option-based measure is that the typical residential borrower is not financially astute, as sophisticated financial models are not freely accessible for the general public. A mortgagor calculates his/her amount of savings (true and ultimate measure of incentive to prepay his/her mortgage!) with refinancing, using currently available alternative mortgage rates and a straight-forward ‘simple’ financial model (2.1) (like in Richard and Roll [40], from which the authors get \( m^0/m^t \) as an appropriate incentive measure). Such “refinancing” calculators are very popular, easily accessible on the world wide web and are often used as an advertisement for refinancing. This may introduce a bias in prepayment behavior of mortgagors which may work in favor of the MRB approach.

Note on close-to-maturity mortgages. The option-based approach can be well-suited for close-to-maturity mortgages since borrowers can use low short interest rate situation, (the mortgage rates can still be high) to repay their mortgages using their own savings or refinancing to balloon or adjustable-rate mortgages.

### 3.2 New approaches

#### 3.2.1 Endogenous MRB

The empirical MRB approach, as implied by its name, is not completely modeled-based because of the exogenous mortgage rate specification. As a result, the model is not immune to structural changes. We cannot tell (predict) the prepayment behavior for different interest rate scenarios until it really happened. To overcome that we can define the mortgage rate process endogenously through the equation (2.11).

The note on the prepayment intensity calibration made in section 3.1.2 is valid for endogenous MRB as well. There is no need for additional calculations of refinancing incentives as opposed to option-based models (except reduced ones). The only complication is valuation of mortgage securities themselves, since in order to do that we have to solve the mortgage rate equation (2.11).

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5Actually, it is more precise to say that the traditional option-based approach assumes that a mortgage is refinanced directly from a bank account in a lump sum. See section 3.2.2 on the topic.
The study of the mortgage rate equation (2.11) is a subject of future research. Solution of (2.11) is numerically expensive and work in this direction is highly desirable. The author tried one version of possible iteration schemes, which is described in section 5.4, and it worked well.

### 3.2.2 One Flaw of the Traditional Option-Based Approach

One of the advantages of the traditional option-based approach is that it depends only on the contract mortgage rate. The current refinancing mortgage rate does not appear in the definition of the prepayment process as it did in the MRB approach. Therefore there is no necessity to define mortgage rate processes before we price mortgage securities itself. But the assumption that made this simplification possible, i.e., the refinancing incentive is the difference of the liability $L_t$ and the outstanding principal $P(t)$, has its flip side. When the borrower refinances he/she repays the outstanding principal with a new mortgage. Therefore he/she assumes a new liability (whose evaluation is based on the new refinancing mortgage rate $m_t$). This point is completely missed in the current literature.

Let us take a look at the definition of the liability at the origination of the mortgage:

\[
L_0 = \mathbb{E} \left[ \int_0^T (c_s + [P_s + F_s] \gamma_s) e^{-\int_t^s [r_s + \gamma_s] d\theta} \, ds \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \int_0^T (c_s + P_s \gamma_s) e^{-\int_t^s [r_s + \gamma_s] d\theta} \, ds \bigg| \mathcal{F}_t \right] + \mathbb{E} \left[ \int_0^T F_s \gamma_s e^{-\int_t^s [r_s + \gamma_s] d\theta} ds \bigg| \mathcal{F}_t \right] = M_0 + \mathbb{E} \left[ \int_0^T F_s \gamma_s e^{-\int_t^s [r_s + \gamma_s] d\theta} ds \bigg| \mathcal{F}_t \right] = P_0 + \mathbb{E} \left[ \int_0^T F_s \gamma_s e^{-\int_t^s [r_s + \gamma_s] d\theta} ds \bigg| \mathcal{F}_t \right] > P_0,
\]

where we used the fact that the price of the mortgage security equals the initial principal. We can draw a conclusion from this that repaying the outstanding principal $P(t)$ with another loan (borrowing the amount of $\tilde{P}_0 = P(t)$), the borrower assumes a new liability which is higher than $P(t)$. Therefore option-based comparison of the current liability with the outstanding principal $P(t)$ is not quite adequate for mortgage valuation.

To correct the option-based approach we should consider a new borrower’s liability instead of $P(t)$ in the definition of $\Pi_t = L_t(m^0) - P(t)$, i.e., we should augment (see (3.3)) the outstanding principal $P(t)$ with the price of the potential re-refinancing transaction cost (the complete option-based approach). The new liability $L_t(m^t)$ depends on the current mortgage rate $m^t$. Here we assume that the new liability is evaluated over the same time period $[t, T]$ as the old one, i.e., the only difference between them is that they are evaluated with the different contracted mortgage rates. It could be more appropriate to consider the new liability over the original term of the mortgage, i.e., $[t, T + t]$, but we are mostly interested providing justification for the practical, reduced option-based approach in the next section, where conclusions would be the same for both specifications of the new liability.

**Note on overestimation of transaction costs.** If we take into account the above consideration and compute refinancing incentive comparing the current and the new liabilities (not just the current liability and the outstanding principal), then a threshold of profitability becomes higher (now the liability should step over a quantity which is higher then the outstanding
principal, see (3.3)), i.e., the current liability must be more expensive to refinance with a profit than it would otherwise be predicted by the traditional option-based approach. That is one of the reasons why the transaction costs implied by the traditional option-based approach in the literature must be very high to explain the prices of mortgage-backed securities on the market.\footnote{See section 4.1.2 for other reasons why current mortgage models predict a long lag in refinancing decision making of mortgagors.}

Since the new liability depends on the current mortgage rate, in order to proceed we must be able to model the mortgage rate process. This, as in the case of the MRB approach, can be done empirically or the process can be deduced endogenously as a solution of the mortgage rate equation (2.11). The latter, of course, greatly complicates the mortgage model.

### 3.2.3 Reduced Option-Based Approach

If a researcher wants to elude mortgage rate modeling (and, as some researchers report, the traditional option-based approach, which has no such a problem, gives promising results), then it appears to be reasonable to use the mortgage price (as an asset to lender) difference as a proxy for refinancing incentive rather than the difference of the liability and the outstanding principal. Indeed, let us consider the “error” of the former approach and compare it with the “error” of the latter approach. In the complete option-based models we have $\Pi_t = L_t(m^0) - L_t(m^t)$, while in the reduced option-based approach $\Pi'_t = M_t - P(t)$ (where the superscript “r” stands for “reduced”). Thus their difference is

$$
\Pi_t - \Pi'_t = (L_t(m^0) - L_t(m^t)) - (M_t - P(t))
$$

where we used the fact that the price of a mortgage asset originated at time $t$ must be equal to the original loan amount $P(t)$ and the definition of the value of a mortgage asset. Let us assume that $\gamma_t(m)$ gives the prepayment behavior of some particular mortgagor. Then it is reasonable to assume that $\gamma_t(m)$ is strictly increasing as a function of $m$. Therefore it is easy to see from (3.4) that $m^0 = m^t$ is equivalent to $\Pi_t = \Pi'_t$. Thus the “prepayment boundary” is detected by $\Pi'_t$ correctly (that is not the case for the traditional option-based approach as we shall see later). When $m^0 < m^t$ then $\gamma_t|_{m^0} < \gamma_t|_{m^t}$ by definition of the prepayment intensity (if one has a mortgage with the lower mortgage rate then he/she is less probable to refinance in the future than if he/she had a mortgage with the higher rate in the same situation) and it is easy to show that the right hand side of (3.4) is negative. Therefore $\Pi'_t$ overestimates $\Pi_t$. When $m^0 > m^t$ the similar logic can tell us that $\Pi'_t$ underestimates $\Pi_t$ by the quantity given on the right hand side of (3.4).

On the other hand the difference between $\Pi_t$ and the traditional refinancing incentive $\Pi'_t$ (where the superscript “tr” stands for “traditional”) is

$$
\Pi_t - \Pi'_t = (L_t(m^0) - L_t(m^t)) - (L_t(m^0) - P(t))
$$
\[
\Pi(t) = P(t) - \mathbb{E} \left[ \int_t^T \left( c_s + [P_s + F_s] \gamma_s \right) e^{-\int_t^T (r_s + \gamma_s) dt} ds \bigg| \mathcal{F}_t \right] m = m^t = -\mathbb{E} \left[ \int_t^T F_s \left[ \gamma_s e^{-\int_t^T (r_s + \gamma_s) dt} \right] m = m^0 \bigg| \mathcal{F}_t \right].
\]  

(3.5)

As we see the traditional option-based refinancing incentive \( \Pi^r_t \) uniformly overestimates the refinancing incentive \( \Pi_t \) by the quantity opposite to (3.5). In particular, \( \Pi_t \) and \( \Pi^r_t \) give different "prepayment boundaries."

Having the right hand side of (3.4) smaller then minus (3.5) we could say that the reduced option-based approach is more promising then the traditional one. The answer, of course, depends on a specification of the prepayment intensity \( \gamma_t \). As a rule of thumb (using the definition of \( \gamma_t \), letting the interest rates be zero and assuming the constant transaction costs) we can say that it is likely to happen if refinancing to a new mortgage does not reduce the probability of future refinancing by more then two.

Apart from the promising, better performance, the reduced option-based approach has another important practical advantage: we do not need to evaluate the borrower’s liability, thus the evaluation and calibration problem becomes simpler. If we modeled one particular borrower (e.g., commercial mortgagor), then when we calibrate \( \gamma_t \) we do not need to do additional calculations to evaluate the refinancing incentive \( \Pi_t \) at all! The mortgage securities data are readily available on the market to estimate \( \Pi_t \), while traditional option-based models require simultaneous solution of the liability equation (3.1). However, when modeling a pool we do not have prices of individual mortgages — only the price of the pool. Therefore, in this case, when we calibrate the distribution of heterogeneous characteristics in a pool, we still have to price individual mortgages, although the problem is simpler than in the traditional option-based approach since we do not price liabilities and we can use the pool price for calibration (because it is a weighted average of individual mortgages in the pool).

Note on the choice of refinancing vehicles. The considered above option-based approach assumed that the current liability, evaluated over the time interval \([t, T]\), is compared with the new liability over the same interval \([t, T]\). That is the borrower is refinancing with a new \((T - t)\)-year mortgage. There is no such flexibility in the mortgage market and, as Rajan and Hayre [21] reported, the 30-year fixed-rate mortgage is still a popular refinancing vehicle. Therefore, the new liability should be evaluated over the time interval \([t, t + T]\), i.e., the \(T\)-year mortgage is refinanced with a new \(T\)-year mortgage. Nevertheless the similar conclusions about the comparison of the traditional and reduced option-based approaches can be drawn in these cases too. The topic will be discussed in the next section.

3.2.4 Option-Based MRB Approach

The difference of the MRB and the option-based approaches was already discussed in section 3.1.3. As the example there shows, the option-based approaches are not sensitive to maturity preferences. When a mortgagor analyzes the possibility of refinancing to another mortgage after \(t\) years of his/her mortgage origination, he/she considers the current mortgage rate whose value is influenced, in particular, by behavior of economic variables over the time interval \([t, t + T]\), variables that are not considered in the option-based approaches considered above. To "introduce" the sensitivity of a mortgage model to behavior of economic variables over full mortgage term since the time of refinancing consideration, we use the following idea. Let \(L^T_t(m)\)

\footnote{Although the topic addressed the empirical MRB and the traditional option-based approach it is valid for all considered approaches so far.}
be the mortgagor’s liability (see (3.1)) for a T-year m-percent mortgage originated at time t with initial principal $P(t)$. The borrower’s refinancing incentive $\Pi_{t}$ can be measured as the difference between his/her liabilities $L_{t}^{T}(m^{0})$ and $L_{t}^{T}(m^{t})$, i.e., it is as if he/she compared two alternatives of refinancing to the same type of mortgages he/she had: the first one is a mortgage with the same contract mortgage rate $m^{0}$ and the other is a mortgage with a currently available for refinancing mortgage rate $m^{t}$. Defined in such a way, the refinancing incentive $\Pi_{t}(m^{0}, m^{t}) = L_{t}^{T}(m^{0}) - L_{t}^{T}(m^{t})$ is just a sophisticated function of two mortgage rates $m^{0}$ and $m^{t}$, so this is a version of the MRB model. For $m_{1} > m_{2}$ ($m_{1} < m_{2}$) we have $\Pi_{t}(m_{1}, m_{2}) < \Pi_{t}(m_{1}, m_{2})$ ($\Pi_{t}(m_{1}, m_{2}) > \Pi_{t}(m_{1}, m_{2})$), what is to be expected from an MRB model (recall that the popular researchers’ choice is $\Pi(m^{0}, m^{t}) = m^{0} - m^{t}$ and $\Pi(m^{0}, m^{t}) = 1 - m^{t}/m^{0}$). Making the comparison of the mortgage rates through the definition of liabilities gives this approach important features of option-based approaches such as dependence of refinancing incentives on interest rate volatilities.

Because the definition of $\Pi_{t}$ includes the current mortgage rate $m^{t}$, we must define the mortgage rate in order to price mortgage securities. This can be done exogenously as in the empirical MRB approach (section 3.1.1) or endogenously with the help of the mortgage rate equation (2.11). For the latter approach we could formally implement the following iterative procedure. Let the process $m_{n}^{0}$ be our first guess. Then subsequent iterations are calculated with the help of the following scheme:

$$m_{n}^{t} \xrightarrow{(3.1)} L_{t}^{T}([m_{n}^{s} \geq t]) \xrightarrow{(2.11)} m_{n+1}^{t},$$

where we can hope that $m_{n}^{t}$ converges to the solution of the equation (2.11).

### 3.2.5 Reduced Option-Based MRB Approach. Time-Homogeneous Diffusion State Setting.

If we assume the traditional option-based approach we would put $L_{t}^{T}(m^{t})$ to be equal to $P(t)$. This would remove the dependence on the mortgage rate process as a whole, making the knowledge of the contract rate $m^{0}$ sufficient for mortgage securities valuation. However, $L_{t}^{T}(m^{t})$ is not equal to $P(t)$ and the considerations of section 3.2.3 can be applied here with minor corrections. From that we can conclude that the similar idea of the reduced model may be appropriate for mortgage modeling. Instead of comparing the liability and the outstanding principal we compare mortgage assets. Namely, let $M_{t}^{T}(m)$ be the price of a T-year m-percent mortgage security originated at time t with the initial principal of $P_{0}$. Multiplication by $P(t)/P_{0}$ “makes” the mortgage be originated with the initial principal $P(t)$. Therefore, “reducing” the option-based MRB model we get $\Pi_{t} = P(t)/P_{0} M_{t}^{T}(m^{0}) - P(t)/P_{0} M_{t}^{T}(m^{t})$. Here we can use the fact that $M_{t}^{T}(m^{t})$ does equal to $P_{0}$ and, therefore, the second term in $\Pi_{t}$ is just $P(t)$. This removes dependence of $\Pi_{t}$ on $m^{t}$, thus relieving the problem of mortgage modeling: $\Pi_{t} = P(t)/P_{0} M_{t}^{T}(m^{0}) - P(t)$.

Although we removed the dependence on the mortgage rate process, the problem is still complicated for practical implementation because to calculate $M_{t}^{T}$ we must know $M_{t}^{T}$, $t < s < t + T$. Therefore a solution to the mortgage equation should be found over time interval $[0, \infty)$. To circumvent the problem we assume that all processes are stationary. From this assumption we get $M_{t}^{T}(\omega) = M_{t}^{T}(\omega)$ for any $t$ and $s, \omega \in \Omega$. In particular we have $\Pi_{t} = P(t)/P_{0} M_{t}^{T} - P(t) = P(t)/P_{0} M_{0}^{T} - P(t)$. If a T-year m-percent mortgage is originated at time 0, then the time t price of the mortgage $M_{t} = M_{t}^{T} m^{0}$ in the present setup is

$$M_{t} = E \left[ \int_{t}^{T} \left( c_{s} + P_{s} \gamma_{s} \left( \frac{P(s)}{P_{0}} M_{0} - P(s) \right) \right) e^{-\int_{s}^{T} \gamma_{r} dr} \left( \frac{P(t)}{P_{0}} M_{0} - P(t) \right) ds \right]_{\mathcal{F}_{t}}. \quad (3.6)$$
Now we have an equation which is defined over time interval \([0, T]\). If we adopt the diffusion state set-up of section 2.2.3 then equation (3.6) is equivalent to the following backward parabolic partial differential equation

\[
\frac{\partial M_t}{\partial t} + A M_t - [r + \gamma_t(M_0)] M_t + c_t + P(t) \gamma_t(M_0) = 0, \quad 0 < t < T
\]

\[
M_T(x) = 0, \quad x \in D.
\]

A peculiarity of this equation is that the non-linear term \(\gamma_t(\cdot)\) depends on the unknown function \(M_0\) at the initial time 0 (it is some sort of PDE with delay). An iterative procedure can be employed to find a solution.
Chapter 4

Specification of Prepayment Function

4.1 Continuity of Prepayment Intensity $\gamma_t$

4.1.1 Why Continuous?

If the prepayment intensity function $\gamma_t$ is not continuous (it is discontinuous in Stanton’s model [43], see section 5.1.2 and footnote 3 in particular), then a classical solution to (2.10) does not exist, but the Feynman-Kac representation still holds if we understand a solution to differential equation (2.10) in a generalized sense. While a discontinuity is a surmountable problem from a theoretical point of view, it is known to give troubles (if “neglected”) for numerical calculations such as a reduced convergence rate and spurious oscillations\(^1\) of the numerical solution. One should apply extra effort to overcome the problem. Thus it is worthwhile to observe here that from an intuitive interpretation of the prepayment intensity $\gamma_t$ it is natural to expect the intensity to be a continuous, even Lipschitz continuous, function of the state process $X_t$. Indeed, $\gamma_t$ corresponds to the rate of prepayment, and we use it to account for non-optimality of prepayment behavior, which is mainly based on “human” factors. Continuity of $\gamma_t$ would model a real “hesitating” human being; it states that a “small” change in economic variables leads to a small change in prepayment activity. A local Lipschitz condition would mean that there are no special economic states for the borrower, as this fuzzy boundary (between profitable and non-profitable states) assumption is a manifestation of a person’s inherent behavioral uncertainty.

This observation is backed up by empirical work (see, e.g., Hayre and Rajan [21]), where traditionally the prepayment curve looks like a smoothed step function. A person hesitates when an object he/she wants costs around the price he/she believes is fair. As the price changes favorably, the person is becoming more decisive (cuspy part of the prepayment curve), and when the price is far beyond the point of profitability, it really does not matter a lot.\(^2\)

Needless to say, if the real prepayment function of a borrower is far from being a step-function (which is implied by many academic models, e.g., Stanton [43], Downing, Stanton and Wallace [11], Kau, Keenan and Kim [31]), acknowledging the continuity will improve predictability of prepayment models.

\(^1\)They usually are negligible for valuation purposes but can have significant impact on hedging performance.

\(^2\)In old times people counted objects like this: 1,2,3,...,10, many, very many. It reflects human psychology very well.
4.1.2 Continuity and Transaction Costs

Apart from numerical advantages and better fit which the continuity of $\gamma_t$ promise, it may also explain transaction costs better than a widely used step-function $\gamma_t$. In this section we illustrate this point.

Let us consider a prepayment intensity which is specified as a step-function:

$$
\gamma = \begin{cases} 
\gamma_p & \text{“profitable”} \\
\gamma_o & \text{“not profitable”} 
\end{cases}
$$

Here $\gamma_p > \gamma_o$. The condition “profitable” can be decoded as the difference of refinancing incentive $\Pi_t$ and the transaction costs $F_t$. If we write the prepayment intensity $\gamma_t$ as a function of $\Pi_t$, then the given specification $\gamma_t(\Pi)$ has a jump at $\Pi = F_t$. Look at it the other way: let we have prepayment rates against refinancing incentive $\Pi_t$ data and we fit a step-function $\gamma_t(\Pi)$ to the data. Then let us try to infer the value of transaction costs. We do not have another choice as to state that $F_t$ is equal to value of $\Pi$ where $\gamma_t(\Pi)$ has a jump.

If we consider a smooth increasing $\gamma_t(\Pi)$ then there is no clear cut candidate for transaction costs $F_t$ but we can reasonably assume that $F_t$ is located at transition of “flat” part of $\gamma_t(\cdot)$ to increasing part. This pushes the transaction costs implied by a smooth prepayment intensity lower then the transaction costs implied by a prepayment intensity with a jump (see Figure 4.1.2).

Indeed, let us assume for simplicity that the transaction cost is a constant. We consider the step-function prepayment intensity $\gamma^s(\cdot)$ and the “smoothed” continuous prepayment intensity $\gamma^c(\cdot)$. These functions depend on “profitability” $\Pi_t - F$. Let the refinancing incentive of some particular borrower be given by an $\mathcal{F}_t$-adapted process $\Pi_t$. Then if we calibrate $F^s$ and $F^c$ to fit $\gamma^s(\Pi_t - F^s)$ and $\gamma^c(\Pi_t - F^c)$ to the prepayment pattern of this borrower, we clearly get $F^c < F^s$ which is due to the gradual increase of $\gamma^c(\cdot)$.

Moreover, if we adopt the option-based approach then there is another effect which pushes $F$ even lower. This is due to differences in the definitions of liabilities implied by different transaction costs. To see this we now calibrate $\gamma^s(\cdot)$ and $\gamma^c(\cdot)$ together with the corresponding refinancing incentives $\Pi^s_t$ and $\Pi^c_t$ as implied by the option-based approach. Next, we have

$$
\gamma^c(\Pi_t - \bar{F}^c) = \gamma^c(\Pi^c_t - [\bar{F}^c + (\Pi^c_t - \Pi_t)]) \approx \gamma^c(\Pi^c_t - F^c)
$$
\[ \gamma^s(\Pi_t - \tilde{F}^s) = \gamma^s(\Pi_t^s - [\tilde{F}^s + (\Pi_t^s - \Pi_t)]) \approx \gamma^s(\Pi_t - F^s), \]

where we used symbol “\(\approx\)” to show that the quantities may not be equal but are close in some sense since it may not be possible to calibrate \(F^c\) and \(F^s\) ideally. From here we can conclude that the transactions costs should be calibrated as follows

\[ F^c \approx \tilde{F}^c + (\Pi_t^c - \Pi_t), \quad F^s \approx \tilde{F}^s + (\Pi_t^s - \Pi_t). \] (4.1)

Because we know \textit{a priori} that \(\tilde{F}^s > \tilde{F}^c\), we can conclude that there is the same relation between the corresponding liabilities with step-function and continuous specifications (see the definition of the liability (3.1)) since \(\gamma^s\) and \(\gamma^c\) are fitted to represent the same borrower.\(^3\) This shows that \(\Pi_t^c < \Pi_t^s\) and, therefore, \(\Pi_t^c - \Pi_t < \Pi_t^s - \Pi_t\). This together with (4.1) gives that not only \(F^s > F^c\), but, moreover, that \(F^s - F^c > \tilde{F}^s - \tilde{F}^c\), i.e., the magnitude of difference of the transaction costs in the cases of continuous and step-function prepayment intensities is greater in the option-based approach than otherwise would be predicted by the “gradual increase” of the continuous intensity.

Summarizing we can draw the following conclusion. Let us consider a step-function and a smooth increasing function as candidates for the prepayment intensity function. If we fit it to the same data then the continuous intensity implies lower transaction costs in terms of economic variables (e.g., interest rates) due to a gradual increase of prepayment rates. This difference may even be magnified for the option-based approaches if the transaction cost is expressed in monetary terms. The huge transaction costs reported in the mortgage option-based literature may be attributed in part to the step-function specification. Another major reason was already considered in section 3.2.2 and is based on the traditional option-based approach misspecification of the refinancing intensity.

\textbf{Note on transaction cost heterogeneity.} Reducing the transaction costs to the “realistic” level we may lose it as a tool to generate the burnout effect (see section 2.1.2 and the footnote 2 there in particular). The transaction cost is the major tool\(^4\) to simulate heterogeneity (and, consequently, burnout) in a pool in the option-based mortgage literature. The other natural candidate for the job is the level of the prepayment rate itself (see section 5.3 for an example of such heterogeneity of prepayment functions in a pool). Since the prepayment pattern is different in this case, the interesting and practically important question is magnitude of heterogeneity of transaction costs and prepayment rates. The author’s conjecture is that the prepayment rates are typically heterogeneous in pools while transaction costs are more or less homogeneous and reflect the real monetary transaction costs (as opposed to the huge portion explained as a psychological cost in the mortgage option-based literature; e.g., Stanton \[43\] found that the psychological transaction cost is 34% while the monetary cost is about 7% of the outstanding principal). \(\blacksquare\)

\subsection*{4.2 Seasoning, Prepayment Intensity and Multiple-Stage Decision Making.}

Seasoning is a term used for the empirically observed phenomenon that mortgage prepayment rates rise from very low levels at issue to much higher levels as the mortgages age. This is the rationale for the Public Securities Association (PSA) standard prepayment model, whose base case models mortgage prepayment rates as increasing linear function of time from 0\% CPR\(^5\)

\(^3\)Actually we can guarantee it for the difference \(\tilde{F}^s - \tilde{F}^c\) large enough because the calibration of the intensity functions clearly is not ideal.

\(^4\)To the best of the author’s knowledge it is the only tool.

\(^5\)CPR is a term for conditional prepayment rate (see Fabozzi [15]). In our set-up it can be thought of as the prepayment rate \(\gamma_t\).
at issue to 6% CPR at thirty months and then remaining constant. This rule of thumb is not applicable for serious valuation since the prepayment is a complicated process which (non-linearly) depends on many factors, but it is an idea of what to expect at the origination of a pool of mortgages.

To account for seasoning researchers multiply the prepayment intensity (which reflects dependence of prepayment on relevant factors) by some ad-hoc increasing function. It is not necessarily the PSA curve. The researchers prefer to fit it according to data they use. For example, Hayre and Rajan [21] used a two-step linear function to account for smooth transition from increasing to steady phase of prepayment curve and, thus, to get better fit to data they used. Stanton used scaled “arctan(·)” for the same purpose.

One of the explanations of the seasoning effect is that borrowers tend not to move right after they take a mortgage to buy a house. While this explanation sounds credible, it is applicable mainly for situations when refinancing is not profitable for the borrower. If it is profitable then it is not likely that borrowers will tend not to refinance just because they recently received mortgages. The prepayment rates are much higher for premium pools than for discount pools (i.e., if a pool is at a premium then most of instances of prepayment are attributed to refinancing to another mortgages with the purpose of reducing the monthly payments), therefore it is not clear why the seasoning effect should be noticeable (as in, for example, Richard and Roll [40]) even in the case of favorable for refinancing circumstances. For example, Downing, Stanton and Wallace [11] assumed that seasoning affects only prepayment due to non-financial reasons (the intensity of exogenous prepayments is proportional to “arctan(·)”) and the financial-based part of the prepayment intensity is a step function, i.e., determination of the borrower to refinance his/her mortgage is not affected by how recently the loan was acquired. In this model, since the financially based intensity is of an order higher than intensity of prepayment due to exogenous reasons, once the rates fall enough, the prepayment rates will not expose (noticeable) seasoning. That is in contrast to Richard and Roll [40] where the authors postulated that duration of the seasoning effect explicitly depends on the refinancing incentive. The higher the refinancing incentive the shorter the mortgage seasoning.

In this chapter we offer a new, mathematically based explanation of the seasoning effect which offers a model-based seasoning curve (as opposed to ad-hoc functions used in the mortgage literature) and which has important consequences for the intensity process throughout the life of a mortgage (not just for a short period of time after mortgage origination). The proposed “seasoning” model can reconcile the difference in approaches to model seasoning such as one that was mentioned above (Downing, Stanton and Wallace [11] vs. Richard and Roll [40]).

Let us see closer how a refinancing decision may be made in a real life and what factors can make a mortgage an illiquid asset (i.e., what factors can prevent or postpone the borrower to refinance as soon as he/she wants to). First, we may acknowledge that an average borrower does not check financial news continuously. He/she can encounter favorable news on TV, see an advertisement on-line, read it in a newspaper, or get it by mail. After the borrower gets the good news that his/her mortgage can be profitably refinanced, he/she usually does not rush right away, there is a period of time to think over the possibility. Next, the borrower may wait some period of time if he thinks the mortgage rates will fall even lower to get a better deal. As we see there are several, presumably conditionally independent, reasons why the borrower will not refinance as soon as it is profitable to do so.

In the case of refinancing due to exogenous reasons (like a new job in another area, divorce, etc.) the borrower can have a series of stages too. First, the timing of a reason to prepay is stochastic. Then the borrower needs time to think over the situation and respond. Finally, as it was noted by many researchers, we can consider the possibility that borrowers are reluctant to move in any circumstances for some (stochastic) initial period of time after the home is purchased.

In either case (the mortgagor considers prepayment due to financial or non-financial reason) it appears that the refinancing process is a complicated multi-staged process which seems to be
reasonable to model with a number of several independent stochastic times.

Let $\tau_1$ and $\tau_2$ be the first jumps of two independent Poisson processes with the constant intensities $\lambda_1$ and $\lambda_2$, i.e., $\tau_1$ and $\tau_2$ are i.i.d. exponential random variables. We consider a two-step decision making process. A person has to complete a task. First, a person waits for a signal to act which arrives stochastically with intensity $\lambda_1$. At time $\tau_1$, i.e., having received the signal, the person needs additional time $\tau_2$ (with expected waited time of $1/\lambda_2$) to actually perform the task. Therefore the time $\tau$ when the task is completed is $\tau_1 + \tau_2$. Let us evaluate the distribution of the stopping time $\tau$. At first we assume $\lambda_1 \neq \lambda_2$.

\[
P(\tau_1 + \tau_2 \leq t) = \int_0^t P(\{\tau_1 \leq s\} \& \{\tau_2 = t - ds\}) = \int_0^t P(\tau_1 \leq s)P(\tau_2 = t - ds)
\]

\[
= \int_0^t (1 - e^{-\lambda_1 s})\lambda_2 e^{-\lambda_2 (t-s)} ds = 1 - \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}.
\]

Now we can compute the intensity $\lambda(t)$ of the random time $\tau = \tau_1 + \tau_2$ (see the definition of the intensity in section 2.2.1):

\[
\lambda(t) = \frac{d}{dt} \ln(1 - P(\tau \leq t)) = \frac{e^{-\lambda_2 t} - e^{-\lambda_1 t}}{\lambda_2 e^{-\lambda_2 t} - \lambda_1 e^{-\lambda_1 t}}.
\]

(4.2)

Analyzing this function we can find that the intensity starts at zero and then gains value and approaches the horizontal asymptote $\lambda = \min\{\lambda_1, \lambda_2\}$. An example of the intensity function $\lambda(t)$, computed for $\lambda_1 = 1$ and $\lambda_2 = 1.5$, is graphed in Figure 4.2. The shape has strong similarity with the ad-hoc “arctan(·)” in Downing, Stanton and Wallace [11] and the Figure 8.4 in Hayer and Rajan [21] where the authors “cut” the angle of PSA prepayment curve to get better fit to the data.

If we consider a three-step random time, i.e., $\tau = \tau_1 + \tau_2 + \tau_3$, we get similar results. Leaving lengthy but straightforward calculations out we just give a formula for the intensity of $\tau$ in the case of (different) constant intensities $\lambda_1$, $\lambda_2$ and $\lambda_3$ for corresponding random times $\tau_1$, $\tau_2$ and $\tau_3$:

\[
\lambda(t) = \frac{(\lambda_2 - \lambda_3) e^{-\lambda_1 t} + (\lambda_3 - \lambda_2) e^{-\lambda_2 t} + (\lambda_1 - \lambda_2) e^{-\lambda_3 t}}{(\lambda_2 - \lambda_3) \lambda_1 e^{-\lambda_1 t} + (\lambda_3 - \lambda_1) \lambda_2 e^{-\lambda_2 t} + (\lambda_1 - \lambda_2) \lambda_3 e^{-\lambda_3 t}}.
\]

An example of the three-step intensity function $\lambda(t)$, computed for $\lambda_1 = 1$, $\lambda_2 = 1.2$ and $\lambda_3 = 6,$
is graphed in Figure 4.3. The difference with two-step intensity is that now the function has “S”-shaped form.

**Note on overestimated transaction costs.** The flat start of the prepayment intensity in the three-stage case has an effect of “transaction costs;” it pushes the intensity curve into the right direction making it longer (i.e., “harder”) to get to the same level of intensity as implied by the two-step prepayment time. This may contribute to a number of reasons already considered in the manuscript why the current option-based mortgage models predict extremely high transaction costs of prepayment.

In the case when \( \lambda_1 = \lambda_2 = \lambda \), i.e., \( \tau_1 \) and \( \tau_2 \) are i.i.d. exponential random variables, \( \tau = \tau_1 + \tau_2 \) has the well known gamma distribution. The intensity of \( \tau \) in this case is given by the following simple formula

\[
\lambda(t) = \left(1 - \frac{1}{1 + \lambda t}\right) \lambda.
\]

In practice, for \( \lambda_1 \approx \lambda_2 \), we can use the above formula as an approximation of (4.2) for \( \lambda = (\lambda_1 + \lambda_2)/2 \) or we can consider additional terms in the expansion of the original formula (4.2) around, say, \((\lambda_1 + \lambda_2)/2\) if \( \lambda_1 \) and \( \lambda_2 \) are not sufficiently close.

To apply the model to mortgage prepayment, we, depending on the kind of intensity (endogenous or exogenous prepayment) we wish to model, identify a number of stopping times and related parameters. The functions we get for endogenous and exogenous parts of prepayment intensity may be quite different (seasoning in *exogenous* prepayment can be prolonged due to fresh recall of “joy” of moving) but they are similar in that both start at zero. Therefore, while origination of the mortgage itself influences only the exogenous intensity (as is reflected in the prepayment rate specification of Downing, Stanton and Wallace [11]) the “seasoning” effect is observable in endogenous prepayments as well (as is implied in Richard and Roll [40]). Thus, even in a recently originated premium mortgage pool,\(^6\) data can show presence of “seasoning” which is a consequence of multi-staged decision making. On the other hand, it is important to see that if the pool was discounted for some time and then became premium, we should see the same type of “seasoning” effect which actually does not depend on how the pool is seasoned.

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\(^6\)Premium refers to a pool where the contract mortgage rate is higher than the current mortgage rate. Discount refers to a pool where the contract mortgage rate is lower than the current mortgage rate.
4.2. SEASONING, PREPAYMENT INTENSITY AND MULTIPLE-STAGE DECISION MAKING

We considered a model with constant intensities of underlying stopping times. The real mortgage model should consider $\mathcal{F}_t$-adapted intensities. For example, $\tau_1$ can refer to arrival of information on profitable opportunities and thus should depend on economic variables (profitable/not profitable), and $\tau_2$ can be the “waiting period,” i.e., may have a constant intensity (depending on a person) or $\mathcal{F}_t$-adapted one as well (i.e., the length of the waiting period is influenced by the state of the economy). The generalization of “$\tau = \tau_1 + \tau_2$” is done in the note at the end of this section. However, the problem here is that summation is not precisely what we expect in (endogenous) refinancing rate modeling in a stochastic environment. Namely, if there was an opportunity for a borrower to prepay the mortgage profitably and it was missed (time $\tau$ corresponds to non-favorable for refinancing time for a given outcome), the “intensity clock” should be “rewound” for that particular borrower (and launched right away or wait until mortgage option will be in-the-money again, what is absolutely the same since the financial intensity is zero if the prepayment option is out-of-the-money). We should keep track from where to start to count time $\tau_1 + \tau_2$, i.e., where increasing functions (like one in Figure 4.2) starts to generate a new stopping time $\tau_1 + \tau_2$ which will generate a new refinancing attempt for that particular borrower.

![Sample intensity of the prepayment time](image)

**Figure 4.4:** Sample intensity of the prepayment time $\tau = \tau_1 + \tau_2 + \tau_3$ conditioned on given information $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The intensities of $\tau_i$ ($i = 1, 2, 3$) are constants.

Complete specification of a “multi-staged” financial (i.e., endogenous) prepayment intensity is a matter of future research. Here we make a somewhat simplifying assumption that borrowers watch the market closely from the point when they see a possibility to refinance their mortgages, i.e., after time $\tau_1$. In this case if refinancing incentive is higher than the transaction costs, then the intensity grows with time according to the formula (4.2) (but may go up or down if refinancing incentive goes up or down depending on specification of the function $\gamma(\cdot)$) as was noted above. As soon as the refinancing opportunity is gone, the borrower acknowledge it and returns to his/her usual state, i.e., the intensity drops back to zero. When the opportunity comes back the intensity starts growing again from the beginning. That is, $\tau_1$, if it happened to be generated for a particular borrower before, is discarded and generated anew from this point in time (or, what is the same, from the time when intensity dropped down to zero). See the sample intensity of prepayment time in Figure 4.4.

**Note on underestimation of prepayment intensity.** From this “seasoning” effect we can conclude that fitting the mortgage prepayment data to a mortgage model with a constant intensity $\lambda$ we inevitably underestimate the real value of the intensity, i.e., the
CHAPTER 4. SPECIFICATION OF PREPAYMENT FUNCTION

estimated average borrower’s waiting period before refinancing (which is 1/λ) is longer than the “physical” one. This may be one of the possible reasons why mortgage models give seemingly unreasonable values for the intensity. For example, Stanton [43] estimates the average waiting period for a borrower to be around two years.

Note on $\mathcal{F}_t$-adapted intensities. Let $\tau_1$ and $\tau$ be stopping times with a $\mathcal{F}_t$-intensity $\lambda^1_t$ and a $\{\sigma(\{s \leq t\} \land s \leq t) \lor \mathcal{F}_t\}$-intensity $\mathbb{1}_{\{t > \tau_1\}} \lambda^2_t$, where $\lambda^2_t$ is a $\mathcal{F}_t$-progressively measurable process. Such a definition gives us in particular $\lambda = \lambda^1_t$. The random time $\tau$ constructed in this way is a summation of random times $\tau_1$ and $\tau_2 = \tau - \tau_1$ where the $\mathcal{F}_t$-intensity process of $\tau_2$ is $\lambda^2_t$ conditioned on $\{\tau_1 < t\}$. We assume that $\tau_1$ and $\tau_2$ are conditionally independent.

Our purpose is to find a $\mathcal{F}_t$-intensity $\lambda_t$ of the random time $\tau$. First, the distribution of $\tau$ is

$$P(\tau \leq t|\mathcal{F}_t) = \int_0^t P(\{\tau_2 \leq t - s\} \land \{\tau_1 = ds\}|\mathcal{F}_t)$$

$$= \int_0^t P(\tau_2 \leq t - s|\{\tau_1 = ds\} \lor \mathcal{F}_t)P(\tau_1 = ds|\mathcal{F}_t)$$

$$= \int_0^t (1 - e^{-\int_0^s \lambda^1_{s'} ds'}) \lambda_t e^{-\int_0^t \lambda^2_{s'} ds'} ds = 1 - e^{-\int_0^t \lambda^2_{s'} ds'} - \int_0^t e^{-\int_0^s \lambda^1_{s'} ds'} ds.$$ 

Then the $\mathcal{F}_t$-intensity of $\tau$ is

$$\lambda_t = -\frac{d}{dt} \ln(1 - P(\tau \leq t|\mathcal{F}_t)) = \frac{\lambda^2 t \int_0^t \lambda^1_s e^{-\int_0^s \lambda^2_{s'} ds'} ds - \int_0^t \lambda^1_s e^{-\int_0^s \lambda^2_{s'} ds'} ds}{e^{-\int_0^t \lambda^2_{s'} ds'} + \int_0^t e^{-\int_0^s \lambda^2_{s'} ds'} ds}.$$

Note on Markovian specification. With this specification of the prepayment time $\tau$ the prepayment intensity is not a Markovian process. In particular, we cannot use the setup of section 2.2.3 directly to find the mortgage price with the help of a PDE. Even with constant $\lambda_1$ and $\lambda_2$ we must keep track of how long ago the borrower’s option was in-the-money because the intensity depends on this time-counter. To be back to the PDE setting we must include this counter into the model as a Markovian process.

Instead of doing this it may turn out to be useful to make the endogenous prepayment intensity $\gamma_t$ depend on an extra variable which tracks not merely the time-counter but “history” of refinancing opportunities as summarized by the following process:

$$h_t = \int_0^t \gamma_s(h_s)e^{-\alpha(t-s)} ds,$$

where $\alpha$ is an empirically defined constant. A history of “missing opportunities” can make the borrower be more careful and faster, therefore we can expect $\gamma_t(h)$ to be an increasing function of $h$.

The process $h_t$ can be defined as a solution of the following differential equation

$$dh_t = (\gamma_t(h_t) - \alpha h_t)dt.$$

Therefore $h_t$ is a Markovian process and we can express the mortgage price with this specification of the process $h_t$ as a solution of a partial differential equation.
5.1 Intensity-Based Version of Stanton’s Model

5.1.1 Review of Stanton’s Model

In Stanton’s model [43] a borrower is assumed to find refinancing profitable when his/her liability is higher than the outstanding principal plus the refinancing cost. The transaction cost should be understood in a wide sense, meaning it includes monetary as well as psychological costs (convenience to go to a bank, fill out forms, time spent on it, etc.). The transaction cost is assumed to be proportional to the principal, i.e., it is \( P(t)F \), and the total borrower’s expenditure in the case of prepayment at time \( t \) is \( P(t)(1+F) \), where \( F \) is an exogenous constant.\(^1\) In addition, the borrower is assumed to “check financial news” (i.e. check if it is profitable to refinance) not continuously but at discrete stochastic time intervals, which are modeled as jumps of a Poisson process with the intensity \( \rho \). At the same time the borrower is exposed to risks of terminating the mortgage prematurely due to exogenous reasons such as divorce, relocation, etc. This is modeled as a jump of another independent Poisson process with intensity \( \lambda \). Finally, Stanton assumes that the risk-neutral dynamics of the interest rate process \( r_t \) can be described by a one-factor, Cox, Ingersoll and Ross model [6], i.e.,

\[
    dr_t = a(b - r_t)dt + \sigma \sqrt{r_t}dW_t. \tag{5.1}
\]

Using econometric considerations Stanton [43] comes up with the following two-step procedure to price a mortgage. The first step is to determine the borrower’s behavior. In the present setup this problem is equivalent to finding the borrower’s liability \( L_t(r) \). In order to find it, Stanton divides the life time of the mortgage into 360 intervals (the number of months in 30 years), each of length \( \Delta t = 1/12 \). He implements the following algorithm to find \( L_{n\Delta t} \) given \( L_{(n+1)\Delta t} \). First, the coupon bond backward PDE is solved (using the Crank-Nicolson method) over the current time step \([n\Delta t, (n+1)\Delta t]\):

\[
    \frac{\partial \hat{L}}{\partial t} + \sigma^2 \hat{L} \frac{\partial^2 \hat{L}}{\partial r^2} + a(b - r) \frac{\partial \hat{L}}{\partial r} - r\hat{L} + c = 0 \tag{5.2}
\]

with the terminal condition \( \hat{L}_{(n+1)\Delta t}(r) = L_{(n+1)\Delta t}(r), \ r \in \mathbb{R}^+ \). The solution \( \hat{L}_{n\Delta t}(r) \) (or \( \hat{L}_n \) for short) of this equation is the borrower’s liability conditional on the prepayment option remaining unexercised over the interval. Then the “true” liability \( L_n \) is found as an expectation of prepay/continue outcomes, i.e.,

\(^1\)Pricing mortgage pass-through securities, Stanton assumes that the transaction cost varies across the borrowers in a pool to model the burnout effect.

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\[ L_n = (1 - Pr(\hat{L}_n))\hat{L}_n + Pr(\hat{L}_n)P(n\Delta t)(1 + F), \]

where \( Pr(\hat{L}) \) is a probability of prepayment over the time interval \( \Delta t \) and is defined as

\[
Pr(\hat{L}) = \begin{cases} 
1 - e^{-\lambda \Delta t} & , \hat{L} \leq P(t)(1 + F) \\
1 - e^{-(\lambda + \rho) \Delta t} & , \text{otherwise.}
\end{cases}
\]

The second main step is to evaluate the mortgage itself. Now prepayment probabilities are known and the mortgage price is found along the same lines as the liability in the first step (we have to change \( L_t \) to the value of the mortgage \( M_t \)) with the only difference being that in formula (5.3) there will not be \( F \) (indeed, the investor receives just the outstanding principal \( P(t) \) upon prepayment, not \( P(t)(1 + F) \)).

### 5.1.2 Option-Based Continuous Time Model

In this section we "translate" Stanton’s assumptions to our framework. There are two sources of information assumed in the model: the interest rate and the timing of prepayment. Therefore the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) can be defined as the natural filtration of the process (5.1).

Recall that Stanton assumes that borrowers prepay in an optimal manner given they know that it is profitable to do so (taking into account transaction costs). Therefore the intensity of the random time, which models the borrower’s refinancing decision, is \( \rho \Delta_t \mathbb{1}_{(L_t(r_t) \leq P(t)(1 + F))} \), where \( L_t(r) \) is the borrower’s liability as a function of the interest rate \( r \) and time \( t \). The other relevant random time, which models prepayment for exogenous reasons, has constant intensity \( \lambda \). The prepayment time is constructed as a minimum of the two independent random times. As is well known, the intensity of the minimum of two independent random times, which coincide with probability zero, is just the summation of their intensities (see, e.g., Bielecki and Rutkowski [2]). Therefore we can define the intensity of the prepayment time \( \gamma_t \) as follows:

\[
\gamma_t = \begin{cases} 
\lambda & , L_t(r_t) \leq P(t)(1 + F) \\
\lambda + \rho & , \text{otherwise.}
\end{cases}
\]

To find \( L_t(r) \) (and, consequently, to determine the prepayment behavior) we need to solve equation (2.10),\(^3\) which in the present setup is

\[
\frac{\partial L}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 L}{\partial r^2} + a(b - r) \frac{\partial L}{\partial r} - [r + \gamma(L)]L + c + P(1 + F)\gamma(L) = 0. \tag{5.5}
\]

\[ L_T(r) = 0, \ 0 < r < \infty. \]

After we have found the liability function \( L_t(r) \), the prepayment intensity (as a function of interest rate) is known and defined by (5.4). The Feynman-Kac representation for the pricing formula (2.4) gives us the PDE for the mortgage price \( M_t(r) \) as a function of the given interest rate \( r \) and the time \( t \):

\[
\frac{\partial M}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 M}{\partial r^2} + a(b - r) \frac{\partial M}{\partial r} - [r + \gamma(L)]M + c + P\gamma(L) = 0, \tag{5.6}
\]

\[ M_T(r) = 0, \ 0 < r < \infty. \]

\(^2\)The second step, actually, is done each time interval right after the first step.

\(^3\) The discontinuity in the specification of \( \gamma_t \) does not allow, in general, for the existence of a classical solution, as we already discussed. We refer the reader to §5.3 of Friedlin [17] on this topic, where one can find a proof of existence of such a solution and some results on its smoothness. However, the specification of \( \gamma_t \) implied by (5.4) put the Stanton’s set-up back into the framework of section [§]. See the last paragraph in the next section for details.
5.1. INTENSITY-BASED VERSION OF STANTON’S MODEL

5.1.3 Relation Between Stanton’s Model and Its Continuous Counterpart

Using the idea of the splitting-up numerical method (see Marchuk [34] or [35] for details), we can apply the following procedure to solve equation (5.5) numerically. Let \( \Delta t \) be a time step and \( L_{n+1} \) be an approximation of \( L_{n+1}\Delta t \). We go backward in time and determine \( L_n \) (the approximation of \( L_{n}\Delta t \)) in two steps.

The first step: we consider the equation

\[
\frac{\partial \hat{L}}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \hat{L}}{\partial \gamma^2} + a(b - r) \frac{\partial \hat{L}}{\partial \gamma} - r \hat{L} + c = 0 \tag{5.7}
\]

with the terminal condition \( \hat{L}_{(n+1)\Delta t} = L_{(n+1)} \). With its help we evaluate the “fractional” step \( L_{n+1/2} = \hat{L}_{n\Delta t} \).

The second step: we consider the remaining part of the original equation (5.5), which is the ODE

\[
L_t + \gamma(L)L + P(1 + F)\gamma(L) = 0 \tag{5.8}
\]

with the terminal condition \( L_{(n+1)\Delta t} = L_{n+1/2} \). Finally, we take \( L_n = L_{n\Delta t} \).

Note on the accuracy of the splitting method. When solving equations (5.7) and (5.8) it is natural to consider only some approximations of the equations since even after exact integration we get just an approximate solution of the original equation. The variant of the splitting-up method we used is of the second order (see section 4.3.1 of Marchuk [34] for a discussion of this “stabilization method.”)

The first step, as we can see, is really the same as the first step in Stanton’s procedure. To see how the second steps are related we explicitly solve the following approximation of (5.8) over the interval \([n\Delta t, (n + 1)\Delta t]\). We freeze the argument of \( \gamma(\cdot) \) at time \( t = (n + 1)\Delta t \), so that it is known over the interval \([n\Delta t, (n + 1)\Delta t]\), i.e., \( \gamma \) is a constant \( \gamma(L_{(n+1)\Delta t}) \). The outstanding principal \( P(t) \) is frozen at time \( t = n\Delta t \). Thus we get the ODE with constant coefficients\(^4\)

\[
\frac{dL}{dt} + \gamma(\hat{M}_{n\Delta t})L + P(n\Delta t)(1 + F)\gamma(\hat{M}_{n\Delta t}) = 0,
\]

which is easy to integrate analytically. Its solution at time \( t = n\Delta t \) is

\[
L_n = e^{-\gamma(L_n)\Delta t}\hat{L}_n + [1 - e^{-\gamma(L_n)\Delta t}]P(n\Delta t)(1 + F). \tag{5.9}
\]

As we can see it is exactly the same as the equation of the second step (5.3) in Stanton’s procedure. Therefore, Stanton’s procedure is a variant of the splitting (fractional step) method for numerical solutions of semi-linear PDE (2.10). It is important to note here that by freezing arguments as above we make (5.9) to be the first order approximation of the solution to the equation (5.8).

Note on the advantage of the splitting method. The splitting method for equation (5.5) that we have considered here is popular, in particular, for numerical integration of reaction-diffusion equations (a non-linear term depends just on the unknown function, not on its derivatives). It is motivated by the fact that the numerical integration of a linear PDE and a scalar ODE are easy, but a numerical integration of the original semi-linear equation involving the two operators together is troublesome. If we opt for an explicit scheme, the time step \( \Delta t \) is limited by \( O(\Delta t^2) \), but if we choose an implicit scheme, we have to solve

\(^4\)Here we use that for the frozen argument of \( \gamma(L_{(n+1)\Delta t}) \) we have \( L_{(n+1)\Delta t} = L_{n+1/2} = \hat{M}_{n\Delta t} \).
a large system of nonlinear equations at each time step, a computationally expensive operation.

Now we can make several observations. Because the prepayment function has a jump, numerical methods, in general, cannot be of the second order approximation unless the discontinuity is specially treated. Additionally, we can get problems with spurious oscillations. However, as one can find, a numerical solution of the traditional option-based approach does not exhibit spurious oscillations while the reduced option-based approach with a step-function $\gamma(\cdot)$ produces notorious oscillations. The reason of such a "strange" phenomena can be seen from the traditional option-based liability equation (in the case of Stanton’s set-up it is equation (5.5)) if we arrange the terms as follows:

$$\frac{\partial L}{\partial t} + A L - r L + c - \Phi \gamma(\Phi) = 0,$$

where $\Phi = L - P(1 + F)$ and $\gamma(\cdot)$ is expressed as a function of $\Phi$. Even though $\gamma(\Phi)$ is discontinuous as a function of $\Phi$, the term $\Phi \gamma(\Phi)$ in the traditional option-based liability PDE is continuous because the jump of $\gamma(\Phi)$ is located at zero. Moreover, it is Lipschitz continuous. Therefore, the liability PDE of time-continuous version of Stanton’s model admits a classical solution. That is not true for the reduced option-based mortgage PDE because the jump of $\gamma(\Phi)$ is located at $F > 0$. Researchers should be careful if they employ a jump-specification of $\gamma(\cdot)$ in this case.

Even with a smooth $\Phi \gamma(\Phi)$ Stanton’s computational method is of the first order in time because the second step is of the first order approximation of the original ODE (5.8). Therefore, the method as a whole is of the first order in time. For the reduced option-based model, then he/she can use fully implicit numerical method to reduce negative effect of spurious oscillations caused by the discontinuity of $\gamma(\cdot)$. In addition, instead of working with (5.3) we can work with some approximation of equation (5.8) of an appropriate (second) order (or an asymptotic expansion ($\Delta t \to 0$) of (5.3)) with the benefit of faster calculations since algebraic operations are performed faster on a computer than evaluations of the exponential functions in (5.3) (recall that the splitting-up method is itself of the second order). In summary, knowing the real PDE standing behind an option-based model, we can be more efficient by using the rich literature on numerical analysis. For example, for Stanton’s mortgage model we can employ a two-step splitting method (e.g., [35]) to get the second order in time at little cost of additional computations. For construction of numerical schemes for differential equations with discontinuous coefficients (as in the case of the reduced option-based approach) the reader can consult Marchuk [34].

### 5.2 A Simple Model

A one factor model for interest rates is probably not adequate for long-lived securities such as mortgages. Two factor interest rate models fit data significantly better and, as Chen and Yang [4] found by testing different interest rate models for mortgage pricing, the ability to fit the initial term structure appears to be the most important characteristic of a sound interest rate model. It is common to consider two factor interest rate model for default-free models, but as soon as default is taken into consideration (e.g., a house price process is included, which is usually one factor diffusion process), researchers revert back to a one factor interest rate model (e.g., compare Schwartz and Torous [41] vs. [42]) because of the numerical burden multi-dimensional problems present. Clearly this should degrade performance of the mortgage models.

Here we propose a MRB model with a simple two factor interest rate model which can be implemented while essentially “paying the computational price” of just a one factor model. Inclusion of other factors related to, say, the house prices, is a straightforward exercise. The
5.2. A SIMPLE MODEL

possibilities of better fitting to the data in this setting promise better performance than option-
based approaches with the same numerical burden (e.g., a one factor model for interest rate and a one factor model for house price).

We assume that the mortgage we want to price is a standard 30-year fixed rate mortgage. Let us write the instantaneous interest rate \( r_t \) as a summation of the 10-year Treasury rate \( l_t \) and a slope \( \delta_t \), i.e. \( r_t = l_t + \delta_t \). The same two factors are used by Boudoukh, Richardson and Stanton [3] in their non-parametric approach, where they use the yield on 10-year Treasury notes and the spread between the 10-year yield and the 3-month Treasury bill yield as factors that determine MBS prices. As the authors report, the 10-year Treasury yield has correlation of 0.98 with the mortgage rate (see Table 9.1 and Figure 9.1, [3]), therefore it is a good candidate to be used as a proxy for the mortgage rate. Thus we put in our model \( \gamma_t = \gamma(l_t, t) \). The prepayment function \( \gamma_t \) can depend on other processes too if they are independent of \( \delta_t \).

We assume that the processes \( l_t \) and \( \delta_t \) are independent. Let filtration \( S_t \) be a natural filtration of the process \( \delta_t \) and \( L_t \) be a natural filtration of the process \( l_t \) and the other processes which are supposed to be independent of \( \delta_t \). We put \( \mathcal{F}_t = S_t \vee L_t \). Then from (2.4) the price of the mortgage is

\[
M_t = \mathbb{E} \left[ \int_t^T \left[ c + P_s \gamma_s(l_s) \right] e^{-\int_t^s \gamma_l(l_s) d\theta} ds \ \bigg| \ \mathcal{F}_t \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ \int_t^T \left[ c + P_s \gamma_s(l_s) \right] e^{-\int_t^s (l_s + \delta_s + \gamma_l(l_s)) d\theta} ds \ \bigg| \ S_t \right] \bigg| \ L_t \right] \\
= \mathbb{E} \left[ \int_t^T \left[ c + P_s \gamma_s(l_s) \right] \mathbb{E} \left[ e^{-\int_t^s \delta_s d\theta} ds \ \bigg| \ S_t \right] \bigg| \ L_t \right].
\]

The expectation \( \mathbb{E} \left[ e^{-\int_t^s \delta_s d\theta} ds \ \bigg| \ S_t \right] \) is the price \( \tilde{B}_s \) of a “bond” in a hypothetical world with “interest rate” \( \delta_t \). For many models (e.g., Vasicek) a closed form expression for \( \tilde{B}_s \) is known, say of the form \( \tilde{B}_s = q(t, s, \delta_t) \). Therefore we can implement a two factor model of interest rates in the present framework by “paying the price” of just a one factor model:

\[
M_t = \mathbb{E} \left[ \int_t^T \left[ c + P_s \gamma_s(l_s) \right] q(t, s, \delta_t) e^{-\int_t^s (l_s + \gamma_l(l_s)) d\theta} ds \ \bigg| \ L_t \right].
\]

Conjecture. These calculations cannot be done in the case of the option-based approach since \( \gamma_t \) depends on \( M_t \) which by definition depends on both processes \( l_t \) and \( \delta_t \). However, knowing that correlation of the mortgage prices vs. the 10-year Treasury yield \( l_t \) is higher than correlation vs. instantaneous interest rate \( r_t \), it is plausible to presume a “weak” dependence between processes \( \gamma_t \) and the slope \( \delta_t \). That leads to the analogous one factor approximate equation for a borrower’s liability

\[
L_t = \mathbb{E} \left[ \int_t^T \left[ c + (P_s + F_s) \gamma_s(L_s) \right] \tilde{B}_s e^{-\int_t^s (l_s + \gamma_l(L_s)) d\theta} ds \ \bigg| \ L_t \right].
\]

This is a one factor interest rate equation with the term \( \tilde{B}_s \) that corrects “the cash flow” with attention to the way the yield curve is humped or sagged. This equation may give a good approximation to the original two factor equation and promises to give better results than those in the literature for option-based models that use a one factor interest rate model.
5.3 CMO Valuation

In this section we propose a new way to price collateral mortgage obligations (CMO’s) using the integral representation (2.5). This framework allows one to use a PDE approach that has never been used before for the problem. We consider a simple form of CMO’s. An owner of a mortgage pool issues $N$ tranches. Cash flow from the mortgages is divided as follows. All tranches receive their proportional part of payment. If some borrower prepay then the principal goes to the first tranch and therefore it does not affect the other tranches. After the first tranch is completely paid off, the turn of the next tranch come to absorb unscheduled payments. So, the last tranch is the last to feel prepayment. As we can see the cash flow is complicated even in this simple case and payments of each tranch depend on prepayment of each mortgagor. Dependence on prepayment behavior is magnified in CMO’s compared to plain mortgage pass-through securities.

We assume that all mortgagors can be classified into the following $K$ categories: $\{\gamma_i\} i = 1, ..., K$. First, we assume that $K = N$ (i.e., the number of categories is the same as number of tranches). We will relax this assumption to $K \geq N$ later. For the price of the $i^{th}$ tranch we use notation $M_i^t$, for the time of $i^{th}$ prepayment — random time $\tau_i$, and for the time of prepayment of $i^{th}$ mortgage — random time $\tau^i$. Recall that $e^{-\int_0^t \gamma_i ds} = Q(\tau > t|\mathcal{F}_t)$. Therefore, from (2.5), the price of the $i^{th}$ tranch is

$$M_i^t = P(t) + \mathbb{E} \left[ \int_t^T (m - r_s)P(s)\Gamma_i(s)e^{-\int_t^s r_e d\theta} ds \mid \mathcal{F}_t \right],$$

where $\Gamma_i(s) = Q(\tau_i > t|\mathcal{F}_i)$. Although $\tau^i$ are dependent, it is natural to assume that events $\{\tau^i > t\}$ are conditionally independent given $\mathcal{F}_t$. \footnote{Indeed, if we fix interest rates and other economic variables, then decision of one mortgagor to prepay is not likely to depend on the other mortgagors in the pool.} If so, using elementary probability, we can compute $\Gamma_i(t) = L_i \left( e^{-\sum_{k \in \Lambda} \int_0^t \gamma_k^i ds} 1_{|\Lambda| > N - i} \right)$, where $L_i$ is a linear form whose dimensionality equals the number of distinct sets $\Lambda \subset [1, ..., N]$ such that $|\Lambda| > N - i$, and where $\gamma^i_k$ is the prepayment intensity of the $i^{th}$ borrower (or, what is the same, of the stopping time $\tau^i$). Therefore the price of the $i^{th}$ tranch can be written as follows

$$M_i^t = P(t) + L_i \left( \mathbb{E} \left[ \int_t^T (m - r_s)P(s)e^{-\int_t^s r_e + \sum_{k \in \Lambda} \gamma_k^i ds} ds \mid \mathcal{F}_t \right] \right) 1_{|\Lambda| > N - i} \left( \int_0^t \Lambda \subset [1, ..., N] \right). \tag{5.10}$$

Each expectation in (5.10) is in the Feynman-Kac form and, therefore, we can use the PDE approach to price CMO’s! To the best knowledge of this author it has never been done before. Knowing the slowness of Monte Carlo method, we can expect that the proposed PDE method of CMO valuation can be faster.

Let us see how many expectations we have in $L_i$. The number of distinct $\Lambda \subset [1, ..., N]$ such that $|\Lambda| > N - i$ equals $\sum_{k=N-i+1}^N C_N^k$. Thus, to find $M_i^t$ we have to evaluate $\sum_{k=N-i+1}^N C_N^k$ expectations. For small $N$ and $k$ this is not a problem, but for large, but still reasonable, $N$ and large $i$ this can be an unsurmountable task. If $i = N$ then the number can be evaluated explicitly and is $2^N - 1$. For example, if the number of tranches is $N = 10$ or 40 and we want to find the price of the $N^{th}$ tranch,\footnote{Recall that for now we assumed that the number of tranches is the same as the number of mortgagors in the pool.} then we have to evaluate 1,023 and 1,099,511,627,775 expectations respectively. The good news is that the probability that all (almost all) mortgagors prepay in the pool is low, i.e., the price of the last tranch(es) is close to the price of a coupon bond. Nevertheless, the number of expectations can still be too high.
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To circumvent the problem we impose a structure on the distribution of $\gamma_i^t$:

$$\gamma_i^t \in \{(n+k)\tau_t | k = 0, ..., K-1\}, \quad (5.11)$$

where $n$ is some fixed integer and $\tau_t$ is a base prepayment function. Thus we assume that each of $N$ borrowers in the pool is in one of $K$ categories (5.11). Figure 6.1 illustrates four representative groups of the borrowers’ prepayment incentives in a pool.

This structure of the distribution of prepayment intensities $\gamma_i^k$ leads to a simplification. For example, if we assume $K = N$ (i.e., the magnitudes of the prepayment intensities in the pool are equally distributed from $n\tau_t$ to $(n+K-1)\tau_t$), then, because of the condition $|\Lambda| > N-i$, we have $\sum_{k \in \Lambda} \gamma_i^k = \tau_t \sum_{k \in \Lambda} (n+k)$, where $\sum_{k \in \Lambda} (n+k) \in \{(N-i+1)(n+1+\frac{N-i}{2}), ..., N(n+\frac{N+1}{2})\}$. So we have ”a lot of” like terms and the dimensionality of the linear form $L_i$ reduces significantly. In particular, in the worst case ($i = N$) we have to find $N(N+1+n) - n - 1$ expectations, i.e., complexity is reduced from exponential to quadratic. For $N = 10$ and $n = 2$ or $N = 40$ and $n = 5$, for example, to find the price of the last tranche (and, in fact, all tranches) we need to evaluate 72 and 1,014 expectations respectively.

In the real world a number of mortgages is much higher than a number of tranches $N$. If $N$ is large then this number may be sufficient to represent heterogeneity in the pool. For a small $N$ more categories may be needed to do this with a necessary precision. Say we have $K$ categories of mortgagors, i.e., a set of prepayment intensities is $\{\gamma_i^k | i \in [1, ..., K]\}$. First, we price the same number $K$ of auxiliary tranches as above. Then the prices of the $N$ original tranches are summations of the appropriate auxiliary ones.

Note on the specification of prepayment heterogeneity. In this section we assumed “proportional” distribution of prepayment intensities in a pool. In an intuitive way it can be understood as if for some economic outcome one borrower is $\alpha$ times faster than another then the same proportion keeps for all other possible economic outcomes. One consequence of this assumption is that the transaction costs for all borrowers, implied by their intensities, are the same. This approach to model a pool’s heterogeneity is opposite of what is assumed in the mortgage literature. They assumed that the major source of burnout is a heterogeneity of the transaction costs. Therefore we have to give a reason for our assumption.

As we already mentioned in section 2.1.2, the burnout effect is a product of this heterogeneity. In the academic literature which adopt a model based approach to burnout, researchers assume
that borrowers’ respond with the same “speed” once the profit from prepayment covers all transaction costs. For example for the $i^{th}$ borrower $(i = 1, \ldots, N)$ in a pool of $N$ mortgages:

$$
\gamma^i_t = \begin{cases} 
\gamma_p & \text{profitable for borrower } i \\
\gamma_o & \text{not profitable for borrower } i.
\end{cases}
$$

It is a level of transaction costs that makes borrowers differ from each other. Each mortgagor has his/her own level of profitability. This specification implies, in particular, that if the mortgage rates fall low enough so that all borrowers in a pool find prepayment profitable and then return to its original value then the pool will not experience burnout effect. As empirical prepayment data show this seems not to be the case. We can conclude that there is some heterogeneity in the value of borrowers’ prepayment intensities $\gamma^i_t$. Moreover, as we suggested in the note in section 4.1.2, we can assume that it is the transaction costs that are uniform among mortgagors and the magnitude of intensities $\gamma^i_t$ is what makes pools exhibit the burnout phenomenon. ■

### 5.4 Numerical Scheme for the Mortgage Rate Equation

In this section we assume that the only stochastic process in the model is the instantaneous interest rate $r_t$. We consider a time-homogeneous model. Thus the mortgage rate process $m^i_t$ is merely a function of the interest rate $r_t$. For the right hand side of the mortgage rate equation (2.11) we use notation $A(m)$, where $m = \{m(r)| r \geq 0\}$ is the mortgage rate function. The mortgage rate equation itself can be written as $m = A(m)$ in this case, i.e., a solution to the mortgage rate equation is a fixed point of the operator $A$. This equation can be written as a family of uncoupled scalar equations $m(r) = A(m(r))$, $r \geq 0$, for the traditional or reduced option-based approaches, but it is essentially a functional equation for, say, (non-empirical) MRB approaches. The numerical scheme below is formulated for the latter case, but it is equally applicable for the option-based approaches if one desires to find the mortgage rate as a function of the interest rate.

The refinancing incentive at time $t$ is assumed to be a function of the contract and current mortgage rates, i.e., $\Pi_t = \Pi(m_0, m_t)$. Since it is not likely that a borrower will refinance to a mortgage with the higher mortgage rate we postulate that if $m_0 \leq m_1$ then $\Pi(m_0, m_1) = 0$. We find it natural to assume monotonicity of the mortgage rate, i.e., $m(r_1) \leq m(r_2)$ if $r_1 \leq r_2$. A model that violates this assumption would look “suspicious” since the fall in the interest rates would imply an increase in mortgage rates having all factors the same. Next, we assume that $\Pi(m_0, m_2) \leq \Pi(m_0, m_1)$ for $m_1 \leq m_2$, i.e., the lower the available-for-refinancing mortgage rate the more likely borrowers refinance their mortgages (recall that the prepayment intensity is an increasing function of the refinancing incentive $\Pi_t$). Combining these monotonicity assumptions on $m(\cdot)$ and $\Pi(m_0, \cdot)$ we get that the prepayment intensity of some particular borrower is a decreasing function of the interest rate $r_t$, just as one would intuitively expect. Finally, we assume that the prepayment intensity is a constant\(^7\) $\gamma_o$ for refinancing incentive $\Pi_t$ less than the transaction cost.

From the above assumptions it follows that if one wants to know, say, the value of some mortgage-backed security for a specific interest rate $r^*$, then he/she needs to know only the mortgage rates for the values of the interest rates not greater than $r^*$, i.e., the set $\{m(r)| r \leq r^*\}$. In particular, $m(r^*)$ can be found from the equation (2.11) knowing only the set $\{m(r)| r < r^*\}$. This idea is behind the following numerical scheme.

Let $M = \{r_i| i = 0, 1, 2, \ldots, N\}$ be a grid over the interest rates values and let $\Delta_i = r_i - r_{i-1}$ be its step sizes. Our purpose is to find approximations of $m(r_i)$, $i = 0, 1, 2, \ldots, N$ for which we\(^7\) $\gamma_o$ can be a function of the interest rate $r_t$ and/or time since the origination of the mortgage to model a burnout effect. It does not change the method below.
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5.4. NUMERICAL SCHEME FOR THE MORTGAGE RATE EQUATION

use notation \( m_i \). We do not specify the numerical method for finding the expectations in the mortgage rate equation (2.11). If this method requires the knowledge of \( m(r) \) for intermediate \( r \)'s in the interest rate grid, then we assume that they are defined with the help of some interpolation based on \( M \). Assume that \( r_0 \) is the lowest value of the interest rate (i.e., zero for most models). Therefore \( m_0 \) can be found as a solution of (2.11) with the constant \( \gamma_0 \) instead of the process \( \gamma_t \), i.e., in this case we expect exogenous prepayment only (e.g., sell of the house due to divorce or new job in another city). This solution can be found with the help of an iterative procedure. Let \( m_0^0 \) be our initial guess, then the sequence \( m^{i+1}_0 \), \( n = 1, 2, \ldots \) is determined inductively:

\[
m^{i+1}_0 = A(m^i_0) = \frac{E \left[ \int_0^T r_s P(s) e^{-\frac{T}{i} \int r_s d\theta - \gamma_0 s} ds \middle| r_0 \right]}{E \left[ \int_0^T P(s) e^{-\frac{T}{i} \int r_s d\theta - \gamma_0 s} ds \middle| r_0 \right]},
\]

where \( m^i_0 \) enters the right hand side through the definition of \( P(t) \) only. Now we expect that \( m^i_0 \rightarrow m_0 \) for \( i \rightarrow \infty \). For this (and for all that follow) iteration we can employ various accelerations (such as Aitkins) to get better performance.

From our specification we can conclude that to find \( m_n \) for \( n = 1, 2, \ldots, N \) we need to know only values of \( m_k \), \( k = 0, \ldots, n-1 \). Indeed, for \( r_k \), \( k = n, \ldots, N \) (i.e., \( r_k \leq r_n \)) we have \( m(r_k) \geq m(r_n) \) and, thus, \( \gamma \) is just a constant \( \gamma_0 \) for these values of the interest rate. Therefore we can find \( m_n \) for \( n = 1, 2, \ldots, N \) in order. If we know the first \( n \) values of \( m_n \) then the next mortgage rate \( m_{n+1} \) can be found with the help of the iterative procedure

\[
m^{i+1}_{n+1} = A(m^i_n; \{m_k | k = 0, \ldots, n\})
\]

\[
= \frac{E \left[ \int_0^T r_s P(s) e^{-\frac{T}{i} \left[ r_s + \gamma(m^i_n; \{m_k | k = 0, \ldots, n\}) \right] d\theta} ds \middle| r_{n+1} \right]}{E \left[ \int_0^T P(s) e^{-\frac{T}{i} \left[ r_s + \gamma(m^i_n; \{m_k | k = 0, \ldots, n\}) \right] d\theta} ds \middle| r_{n+1} \right]}.
\]

Finding \( m_n \)'s successively we provide ourselves with a good initial guess for this iterative procedure. From econometric considerations we expect the function \( m(r) \) to be continuous for a reasonable model. Therefore \( m_n \) should not be "far" from \( m_{n-1} \). If the model gives differentiability\(^8\) of \( m(r) \) then we can use Euler’s rule to take the even better guess \( m^i_{n+1} = m_n + (m_n - m_{n-1}) (\Lambda_{n+1}/\Lambda_n) \). This guess significantly reduces the number of iterations needed for a given precision. The reader can find Figure 5.2 helpful for understanding the mortgage rate iterative procedure.

**Note on the interest rate grid.** Suppose the expectations in the mortgage rate equation (2.11) are evaluated on the grid \( \mathcal{R} = \{r_i | i = 0, 1, 2, \ldots, M \} \) with the characteristic step size \( \Delta_{\mathcal{R}} \). Assume that the mortgage rate function \( m(r) \) is twice differentiable and is approximated on the grid \( \mathcal{M} = \{r_i | i = 0, 1, 2, \ldots, N \} \) with the characteristic step size \( \Delta_{\mathcal{M}} \). If the method employed to find the expectations is of the second order then it is reasonable to take \( \mathcal{M} = \mathcal{R} \). However, if the method is of the first order then we can calculate the mortgage rate function \( m(r) \) on the grid \( \mathcal{M} \subset \mathcal{R} \) with the characteristic step size \( \Delta_{\mathcal{M}} = O(\sqrt{\Delta_{\mathcal{R}}}) \) interpolating the values of \( m(r) \) to \( \mathcal{R} \) (for calculation of the expectations) with the help of the second order spline approximation.

The numerical scheme was tested on the interest rate and \( \gamma(\cdot) \) specifications used by Stanton in [43]. We refer the reader to section 5.1.1 where we reviewed this approach. The coefficients

\(^8\) We have twice differentiability, for example, if the model satisfy conditions in section 2.2.3.
of the CIR interest rate model (5.1) are taken as follows:\(^9\)

\[
\begin{align*}
a &= 0.17203 \\
b &= 0.13546 \\
\sigma &= 0.11425.
\end{align*}
\]

As we can see the long run mean is 13.5% under the martingale measure. Taking into account the market price, which is implied to be \(\lambda = -1.06477\sqrt{r}\) in [43], the long run mean is 7.9% under the physical measure. We remind the reader that the prepayment intensity in [43] was assumed to be a step-function

\[
\gamma(\Pi_t) = \begin{cases} \lambda & \text{if } \Pi_t \leq P(t)F \\ \rho + \lambda & \text{otherwise,} \end{cases}
\]

where \(\lambda\) is an exogenous part of the prepayment intensity, i.e., it refers to prepayments due to exogenous reasons (such as new job, divorce, etc.) and \(\rho\) is an endogenous part of the prepayment intensity, i.e., it refers to prepayments due to financially based decisions.

We choose the transaction cost to be 30% of the outstanding principal, i.e., \(F\) in the above specification of \(\gamma\) is 0.3.\(^{10}\)

We consider the traditional option-based approach, i.e., \(\Pi_t = L_t - P(t)\) (exactly as in Stanton [43]) and the endogenous MRB with \(\Pi_t = P(t)c(m^0)/c(m^t)\),\(^{11}\) where \(c(m)\) is a coupon rate as determined by the contracted mortgage rate \(m\) (i.e., \(c(m)\) is a solution of (2.1) and (2.2) for a fixed \(m\)). The results “the option-based vs. MRB approaches” are given on Figure 5.3. We can see that the “S”-shaped graphs of the MRB mortgage rates are quite different from the graphs of the option-based mortgage rates.

The increase of the exogenous part of the prepayment intensity \(\lambda\) smoothes the graphs of both approaches, but the change is more pronounced and is more complex in the MRB graph. This can be seen on Figure 5.4.

The difference is even more striking in sensitivity to variation of the endogenous part of the prepayment intensity \(\rho\). This is illustrated on Figure 5.5.

\(^9\)The interest rate model in Stanton [43] was stated under the physical probability measure, therefore the CIR parameters there are different from ours. Our coefficients and the market price of risk deducted from this parameters and the pricing PDE.

\(^{10}\)Stanton estimated \(F\) to be concentrated in the range of 30 – 50% with a mean value of 41%.

\(^{11}\)The fraction \(c(m^0)/c(m^t)\) shows the proportion of how much is saved with refinancing, i.e., it is a relative quantity. Multiplying by \(P(t)\) we express the savings in terms of an absolute quantity.
Figure 5.3: The mortgage rates as implied by the MRB (solid line) and the option-based (dotted line) approaches. The graphs are given for two different values of $\lambda$: 0 (left graph) and 0.05 (right graph); $\rho = 2$.

Figure 5.4: The mortgage rates as implied by the option-based (left graph) and the MRB (right graph) approaches. The graphs are given for two values of $\lambda$: 0 (solid lines) and 0.05 (dotted lines); $\rho = 2$. 
Figure 5.5: The mortgage rates as implied by the option-based (left graph) and the MRB (right graph) approaches. The graphs are given for four values of $\rho$: 0 (solid lines), 0.6 (dashed lines), 2 (dot-dashed lines), and $\infty$ (dotted lines); $\lambda = 0$. On the option-based graph the cases $\rho = 2$ and $\rho = \infty$ are practically indistinguishable.
Chapter 6

Conclusion.

This thesis developed a general model of default-free mortgages subject to prepayment risk by using an intensity-based approach. The model is flexible and can be applied to all types of mortgages available on the market. Moreover, we showed that our approach generalizes all of the models in the literature. To do this, we classified (parametric) mortgage models into MRB (Mortgage-Rate-Based) and option-based groups due to the way the borrower’s prepayment incentive is measured. This division is natural and is validated by the fact that implementations of MRB and option-based models use different analytical and numerical approaches. We considered different specifications of our model in the view of this new classification.

Our general model is not tied to a particular numerical procedure as are some existing models (e.g., Kau, Hilliard and Slawson [29], Downing, Stanton and Wallace [11]). As an example we showed that Stanton’s model [43] is in fact just a variant of a splitting-up, numerical method applied to our model. Knowledge of the real underlying process can give an edge to a researcher since he/she can be numerically more efficient. For example, Stanton’s approach,¹ which is of the first order of convergence, can be easily “upgraded” to the second order. This is very important for mortgage securities, because mortgage modelling is a computationally heavy problem.

Our study of the prepayment process and refinancing incentive is new and leads to a deeper understanding of the prepayment process of an individual mortgagor and mortgage pools in general.

Throughout the text we pointed out new possible ways to develop mortgage modelling and to raise its efficacy. In forthcoming work we shall develop a mortgage model subject to default as well as prepayment risk. Another topic will be the study of the mortgage rate equation (2.11) and the relation between the mortgage rates implied by different model specifications (MRB and option-based approach) and long term Treasury yields. In the case of MRB specifications we shall study the generalized mortgage equation which will include the possibility of refinancing to different types of mortgages, say 30-year mortgage can be refinanced to 15- or 30-year mortgages. We will also consider numerical approximations of this equation closer.

¹It is the most advanced approach in the academic literature in certain sense.
Cited literature


