On Efficient Implementation of the Option-Based Approach to Mortgage Modeling

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Abstract
In the current literature option-based mortgage models (which recognized “sub-optimality” of prepayment) are essentially discrete-time. In this paper we find a continuous-time limit of a well-known Stanton’s model. We show that Stanton’s model is a version of a splitting-up numerical scheme applied to a semi-linear PDE of this continuous-time limit model. Then we illustrate a computational advantage of continuous-time formulation. In particular, the proposed second-order predictor-corrector numerical scheme gives an acceptable solution in about ten time steps. This is a substantial reduction of computations compared to 360 (number of months in 30 years) time steps often used in discrete-time models.

Key Words: mortgage, option-based approach, prepayment, intensity, finite difference, splitting-up, predictor-corrector, Crank-Nicholson scheme.

1 Introduction
The most important characteristic of a mortgage model is an ability to predict prepayment rates in a mortgage pool. Mortgagors have the option to prepay fully or partially their loans prior to the maturity dates. This right has a dramatic effect on valuation by introducing cash flow uncertainty which depends on the mortgage holder’s view of possible future opportunities (e.g., the mortgage holder’s expectation of the future behavior of the yield curve) to refinance the loan. A number of ideas were proposed in the literature to create a prepayment model, but the problem is yet to get a satisfactory solution.

An important feature of a prepayment model is how one models prepayment incentives (i.e., a borrower’s idea of profitability of prepayment) of the borrower. This determines the borrower’s decision making implied by the model. One group of approaches to model this incentive is closely related to pricing of American options and therefore we will call them the option-based approaches. The option-based measure of prepayment incentive is endogenous. The central object of the approach is a borrower’s liability, which can be defined as the present value of cash flow that the borrower will pay off to repay his/her loan plus, possibly, various transaction costs which are incurred in case of prepayment or default. Current option-based models measure the mortgagor’s incentive to prepay as the difference between this liability and outstanding principal. This assumption somewhat simplifies the problem of mortgage modeling, because it removes the necessity of the mortgage rate modeling, ¹ although it is equivalent to an assumption that borrowers are allowed to refinance their

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¹In general, the endogenous mortgage rate model is computationally quite hard (see Goncharov [5]). In the current literature this problem is solved simply by using 10-year Treasury yield or 10-year swap rate as a proxy for 30-year mortgage rate.
loans only once (see Goncharov [6] for discussion of alternative option-based approaches without this implication). Additionally, the option-based approach is criticized for the "efficient market"-like basis for the refinancing incentive (the borrower's liability is usually priced with the standard risk-neutral argument). This author is currently working on the answer to these problems [7].

The option-based approach has evolved from frictionless models with optimal prepayment behavior (e.g., Dunn and McConnell [3], Kau, Keenan, Muller and Epperson [8]), where the mortgage liability to the borrower and the mortgage asset to the investor are not distinguished (or they differ due to fees only) and the borrowers terminate their mortgages if and only if it is financially optimal, to models which recognize the importance of taking into account the substantial presence of transaction costs and non-optimal behavior (the borrower can fail to prepay when it is financially profitable and he/she can prepay when it is not profitable to do so). Transaction costs were first incorporated in the model as a part of refinancing threshold for optimal liability (e.g., Dunn and Spatt [4]). All these mentioned models assumed optimal prepayment plus "background" prepayment due to relocation, divorce, etc. Stanton [11] first acknowledged the fact that borrowers fail to prepay even if it is optimal to do so. Stanton's model, as opposed to the models above, permits mortgage prices to be higher than the outstanding principal plus the transaction costs (this phenomena is actually observed on market). Recently, Longstaff [9] proposed a model with an optimal borrower's behavior, which, nevertheless, is able to explain this phenomena. The key observation in this paper, which separates it from previous models in the literature, is that borrowers refinance their loans with another loan which might be "expensive" due to credit rating constraints. Independently, a similar idea (but under prepayment sub-optimality assumption) was implemented in the model called "complete option-based" by Goncharov [7].

This paper presents a continuous-time option-based mortgage model in a popular diffusion state set-up. The borrower's prepayment intensity is modeled as a function of the borrower's liability which is a solution of a reaction-diffusion PDE. All current option-based models (which recognize non-optimality of prepayment) can be regarded as discrete first-order approximations to this PDE. A non-trivial question in this context is how option-based models proposed in the literature, in particular Stanton's model, relate to this continuous-time model. We show that Stanton's model is a form of a splitting-up numerical scheme applied to the original reaction-diffusion PDE. Next, we show how the knowledge of this PDE can help to be numerically efficient. In particular, we show that ten time steps are sufficient for computation if a second-order predictor-corrector is used for the underlying PDE. This is a substantial reduction of computations compared to 360 (number of months in 30 years) time steps used by Stanton.

2 Mortgage Model

2.1 Mortgage Model

We consider the following contract. A borrower takes a loan of $P_0$ dollars at some initial time and assumes the obligation to pay scheduled coupons at rate $c$ continuously for duration $T$ of the contract. The loan is secured by the collateral of some specified real estate property, which obliges the borrower to make the payments. The interest on the principal is compounded according to a fixed contract mortgage rate $m$.\footnote{The mortgage rate $m$ and the payments $c$ are assumed constant merely for simplicity. They can be deterministic functions of time (as in graduate-payment or growing-equity mortgages) or stochastic processes (as in adjustable-rate mortgages).} Given, the mortgage rate $m$ and the scheduled payments $c$, the outstanding principal $P(t)$ can be computed.

The borrower has the right to settle his/her obligation during an interval specified by contract\footnote{Commercial mortgages, for example, often have a prepayment lookout period.} and prepay the outstanding principal $P(t)$ in a lump sum. If the borrower refinances then he/she
is forced to pay concomitant transaction costs. The transaction cost process \( F_t \) is assumed to be a defined part of the mortgage model.

Our model will be built around the state variable process \( X_t = (X^1_t, \ldots, X^n_t) \) which is a diffusion process following the stochastic differential equation

\[
(2.1) \quad dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_s = x \in D
\]

with an \( m \)-dimensional Brownian motion \( W_t \in \mathbb{R}^m \) and functions \( \mu : [0, T] \times D \to \mathbb{R}^n, \sigma : [0, T] \times D \to \mathbb{R}^n \times \mathbb{R}^m \), where \( D \) is a domain in \( \mathbb{R}^n \).

We consider the usual information structure described by a natural filtration of the Brownian motion \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q) \). We interpret \( \{\mathcal{F}_t\}_{t \geq 0} \) as a model of the flow of public information which is not borrower-specific. The filtration is an intrinsic feature of the market: this means that all traders have the same information available at any time. The Brownian motion represents the source of randomness of prepayments due the economic factors affecting mortgage rates.

We model prepayment time as a first jump a generalized Poisson process (so called Cox process). This means that the probability of prepayment is driven by some \( \mathcal{F}_t \)-intensity process \( \gamma_t \), i.e., intuitively speaking the intensity of prepayment in a “large” hypothetical pool of “homogeneous” borrowers\(^4\) is \( \gamma_t \) (which is determined by the state of economy \( \mathcal{F}_t \)). Equivalently, a probabilistic interpretation gives \( Q(t < \tau \leq t + \Delta t \mid \mathcal{F}_t \vee \{\tau > t\})/\Delta t \approx \gamma(t) \) for small \( \Delta t \).

We assume that at any time \( t \) it is possible to invest one unit in a default-free deposit account at a short-term interest rate \( r_t \) and “roll-over” the proceeds until a later time \( s \) for a market value at that time of \( e^{\int_t^s r_t \, ds} \).

The processes \( r_t, F_t \) and \( \gamma_t \) are assumed to be deterministic functions of time \( t \) and the state process \( X_t \). Additionally, as was pointed out in the introduction, the option-based approach implies that \( \gamma_t \) also depends on the the borrower’s liability \( L_t \) (defined by (2.3) below). To avoid long expressions we use notation \( f_t \) for \( f(t, X_t) \), so that instead of writing \( r(t, X_t) \) and \( \gamma(t, X_t, L(t, X_t)) \) we use notations \( r_t \) and \( \gamma_t(L_t) \).

It can be shown (see Goncharov [5]) that the mortgage price and the borrower’s liability can be expressed as the following expectations respectively:

\[
(2.2) \quad M_t = \mathbb{E}_t \left[ \int_t^T \left( c + P(s)\gamma_s \right)e^{-\int_t^s (r_s + \gamma_s) \, ds} \, ds \mid \mathcal{F}_t \right].
\]

\[
(2.3) \quad L_t = \mathbb{E}_t \left[ \int_t^T \left( c + [P(s) + F_s]\gamma_s \right)e^{-\int_t^s (r_s + \gamma_s) \, ds} \, ds \mid \mathcal{F}_t \right].
\]

The only difference in the mortgage price and the borrower’s liability is the transaction cost \( F_t \) which is paid the borrower to third party and is not collected by the lender.

**Note on the liability specification.** The liability \( L_t \) is not just the discounted value of the cash flow \( c \); it takes into account future prepayment opportunities. It is important to note the presence of \( \gamma_t \) and \( F_t \) in the definition of the liability, i.e., the borrower is aware of the possibilities to miss prepayment opportunities in the future and takes into account the future transaction costs associated with the prepayment. This feature is sometimes overseen in the option-based prepayment modeling literature. \( \blacksquare \)

Let operator \( \mathcal{A} \) be the generator of the diffusion state process \( X_t \), i.e.,

\[
\mathcal{A} := \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \mu_i \frac{\partial}{\partial x_i}.
\]

\(^4\)That is the speed of prepayment in terms of proportion of the borrowers in this pool per unit of time.
Then under certain technical conditions (see Goncharov [5]) the mortgage price and the borrower’s liability are solutions \( M_t \) of the following backward PDEs:\(^5\)

\[
\frac{\partial M}{\partial t} + \lambda M - [r + \gamma(L)]M + c + P\gamma(L) = 0, \quad M_T(x) = 0.
\]

\[
\frac{\partial L}{\partial t} + \lambda L - [r + \gamma(L)]L + c + (P + F)\gamma(L) = 0, \quad L_T(x) = 0.
\]

Note that PDE (2.5) is non-linear (semi-linear) and PDE (2.4) is linear. After we solve equation (2.5), the prepayment intensity \( \gamma \) is a “known” function and enters PDE (2.4) as a “usual” coefficient.

### 3 Intensity-Based Version of Stanton’s Model

#### 3.1 Review of Stanton’s Model

In Stanton’s model [11] a borrower is assumed to find refinancing profitable when his/her liability is higher than the outstanding principal plus the refinancing cost. Stanton understands the transaction cost in a wide sense, meaning it includes monetary as well as psychological costs (convenience to go to a bank, fill out forms, time spent on it, etc.). The transaction cost is assumed to be proportional to the principal, i.e., it is in our notation \( F_t = P(t)F \), and the total borrower’s expenditure in the case of prepayment at time \( t \) is \( P(t)(1 + F) \), where \( F \) is an exogenous constant.\(^6\) In addition, the borrower is assumed to refinance optimally while “checking financial news” (i.e., checking if it is profitable to refinance) not continuously but at discrete stochastic time intervals, which are modeled as jumps of a Poisson process with the intensity \( \rho \).\(^7\) At the same time the borrower is exposed to risks of terminating the mortgage prematurely due to exogenous reasons such as divorce, relocation, etc. This is modeled as the first jump of another independent Poisson process with intensity \( \lambda \). Finally, Stanton assumes that the risk-neutral dynamics of the interest rate process \( r_t \) can be described by a one-factor, Cox, Ingersoll and Ross model [2], i.e.,

\[
dr_t = a(b - r_t)dt + \sigma \sqrt{r_t}dW_t.
\]

Using econometric considerations Stanton [11] comes up with the following two-step procedure to price a mortgage. The first step is to determine the borrower’s behavior (to estimate “probability” of prepayment given certain information, i.e., to find prepayment intensity).\(^8\) In the present setup this problem is equivalent to finding the borrower’s liability \( L_t(r) \). In order to find it, Stanton divides the life time of the mortgage into 360 intervals (the number of months in 30 years), each of length \( \Delta t = 1/12 \). He implements the following algorithm to find \( L_{n\Delta t} \) given \( L_{(n+1)\Delta t} \). First, the coupon bond backward PDE is solved (using the Crank-Nicolson method) over the current time step \([n\Delta t,(n+1)\Delta t]\):

\[
\frac{\partial \hat{L}}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \hat{L}}{\partial r^2} + a(b - r) \frac{\partial \hat{L}}{\partial r} - r\hat{L} + c = 0
\]

\(^5\)We left out the dependence of \( L, M, r, c, \gamma, P \) on time \( t \) and the value of the state variables \( x \) for notational simplicity and to underline the nonlinear dependence of \( \gamma \) on \( M \) (we will keep this practice in what follows).

\(^6\)Pricing mortgage pass-through securities, Stanton assumes that the transaction cost varies across the borrowers in a pool to model the burnout effect.

\(^7\)Alternatively, it can be viewed as if the borrower “check financial news” more frequently, i.e., with some higher intensity \( \rho' \), but use every refinancing opportunity with some probability \( h < 1 \) so that \( \rho = h\rho' \). This can partially explain the low value of estimated \( \rho \approx 0.6 \), which means that borrowers “check financial news” in about one and a half years on average \( (1/\rho) \).

\(^8\)The object of the first step is prepayment modeling and is most important in a certain sense. The inability to satisfactorily predict prepayment is the main challenge in dealing with mortgage securities.
with the terminal condition \( L_{(n+1)Δt}(r) = L_{nΔt}(r) \), \( r \in \mathbb{R}^+ \). The solution \( \hat{L}_{nΔt}(r) \) (or \( \hat{L}_n \) for short) of this equation is the borrower’s liability conditional on the prepayment option remaining un-exercised over the interval. Then the “true” liability \( L_n \) is found as an expectation of prepay/continue outcomes, i.e.,

\[
L_n = (1 - Pr(\hat{L}_n))\hat{L}_n + Pr(\hat{L}_n)P(nΔt)(1 + F),
\]

where \( Pr(\hat{L}) \) is a probability of prepayment over the time interval \([nΔt, (n+1)Δt]\) and is defined\(^9\) as

\[
Pr(\hat{L}) = \begin{cases} 
1 - e^{-\lambdaΔt} & \hat{L} \leq P(t)(1 + F) \\
1 - e^{(-\lambda + ρ)Δt} & \text{otherwise.}
\end{cases}
\]

The second main step is to evaluate the mortgage itself. Now prepayment probabilities are known and the mortgage price is found along the same lines as the liability in the first step\(^10\) (we have to change \( L_t \) to the value of the mortgage \( M_t \) with the only difference being that in formula (3.3) there will not be \( F \) (indeed, the investor receives just the outstanding principal \( P(t) \) upon prepayment, not \( P(t)(1 + F) \)).

### 3.2 Option-Based Continuous Time Model

In this section we “translate” Stanton’s assumptions to our framework. Recall that Stanton assumes that borrowers prepay with intensity \( ρ \) given they know that it is profitable to do so (taking into account transaction costs). Therefore the intensity of the random time, which models the borrower’s refinancing decision, is \( \rho I[L_t(r_t) \leq P(t)(1 + F)] \), where \( L_t(r) \) is the borrower’s liability as a function of the interest rate \( r \) and time \( t \). The other relevant random time, which models prepayment for exogenous reasons, has constant intensity \( λ \). The prepayment time is constructed as a minimum of the two independent random times. As is well known, the intensity of the minimum of two independent random times, which coincide with probability zero, is just the summation of their intensities (see, e.g., Bielecki and Rutkowski [1]). Therefore we can define the intensity of the prepayment time \( χ_t \) as follows:

\[
χ_t := χ(t, L_t(r_t)) = \begin{cases} 
λ & L_t(r_t) \leq P(t)(1 + F) \\
λ + ρ & \text{otherwise.}
\end{cases}
\]

To find \( L_t(r) \) (and, consequently, to determine the prepayment behavior) we need to solve equation (2.5) which in the present setup is

\[
\frac{∂L}{∂t} + \frac{σ^2}{2} r \frac{∂^2L}{∂r^2} + a(b-r)\frac{∂L}{∂r} - [r + χ(L)]L + c + P(1 + F)χ(L) = 0.
\]

\( L_T(r) = 0, \ 0 < r < \infty \).

After we have found the liability function \( L_t(r) \), the prepayment intensity (as a function of interest rate) is known and defined by (3.4). The Feynman-Kac representation for the pricing formula (2.2) gives us the PDE for the mortgage price \( M_t(r) \):

\[
\frac{∂M}{∂t} + \frac{σ^2}{2} r \frac{∂^2M}{∂r^2} + a(b-r)\frac{∂M}{∂r} - [r + χ(L)]M + c + Pχ(L) = 0,
\]

\( M_T(r) = 0, \ 0 < r < \infty \).

\(^9\)This probability is easy to find using well known properties of Poison processes.

\(^{10}\)The second step, actually, is done each time interval right after the first step.
3.3 Relation Between Stanton’s Model and Its Continuous Counterpart

Let us write the semi-linear liability PDE (2.3) in the following form:

$$\frac{\partial L}{\partial t} + B_1 L + B_2 L = 0,$$

where

$$B_1 := (A - r + c)L, \quad B_2 := (P + F - L)\gamma(L).$$

This PDE can be solved numerically, using the idea of the splitting-up numerical method (see Marchuk [10] for details). The idea is based on an approximation of a solution of (3.7) by solutions of simpler (split) equations

$$\frac{\partial L}{\partial t} + B_1 L = 0, \quad \frac{\partial L}{\partial t} + B_2 L = 0,$$

where the former equation is a linear PDE and the latter is a non-linear ODE.

Let us consider one version of the splitting-up method to find a solution of equation (3.5) numerically. Let \(\{0, \Delta t, 2\Delta t, \ldots, N\Delta t = T\}\) be a uniform time grid and \(L_n\) be an approximation of \(L_{n\Delta t}\). We go backwards in time and given \(L_{n+1}\) we determine \(L_n\) in two steps for every time interval \([n\Delta t, (n+1)\Delta t]\), \(n = N - 1, \ldots, 0\).

The first step: we consider the equation (the first equation in (3.8))

$$\frac{\partial \hat{L}}{\partial t} + \frac{\sigma^2}{2} r^2 \frac{\partial^2 \hat{L}}{\partial r^2} + a(b-r)\frac{\partial \hat{L}}{\partial r} - r\hat{L} + c = 0$$

with the terminal condition \(\hat{L}_{(n+1)\Delta t} = L_{(n+1)}\). With its help we evaluate the “fractional” step \(L_{n+1/2} = \hat{L}_{n\Delta t}\).

The second step: we consider the remaining part of the original equation (3.5), which is the ODE (the second equation in (3.8))

$$\frac{dL}{dt} + (P(1+F) - L)\gamma(L) = 0$$

with the terminal condition \(L_{(n+1)\Delta t} = L_{n+1/2}\). Finally, we take \(L_n = L_{n\Delta t}\).

Thus, given \(L_N \equiv 0\), we can repeat these two steps to find \(L_n\) for all \(n = N - 1, \ldots, 2, 1, 0\).

**Note on the accuracy of the splitting method.** When solving equations (3.9) and (3.10) it is natural to consider only some approximations of the equations since even after exact integration we get just an approximate solution of the original equation. See Marchuk [10] for details.

The first step, as we can see, is really the same as the first step in Stanton’s procedure [11]. To see how the second steps are related, we explicitly solve the following approximation of (3.10) over the interval \([n\Delta t, (n+1)\Delta t]\). We freeze the argument of \(\gamma(\cdot)\) at time \(t = (n + 1)\Delta t\), so that it is known over the interval \([n\Delta t, (n+1)\Delta t]\), i.e., \(\gamma\) is a constant \(\gamma(L_{(n+1)\Delta t})\). The outstanding principal \(P(t)\) is frozen at time \(t = n\Delta t\). Thus we get the ODE with constant coefficients

$$\frac{dL}{dt} - \gamma(\hat{L}_{n\Delta t})L + P(n\Delta t)(1+F)\gamma(\hat{L}_{n\Delta t}) = 0,$$

which is easy to integrate analytically. Its solution at time \(t = n\Delta t\) is

$$L_n = e^{-\gamma(L_n)\Delta t}\hat{L}_n + [1 - e^{-\gamma(L_n)\Delta t}]P(n\Delta t)(1+F).$$

As we can see it is exactly the same as the equation of the second step (3.3) in Stanton’s procedure.

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11Here we use that for the frozen argument of \(\gamma(L_{(n+1)\Delta t})\), we have \(L_{(n+1)\Delta t} = L_{n+1/2} = \hat{L}_{n\Delta t}\).
We can conclude that Stanton's procedure is a variant of the splitting (fractional steps) method for numerical solutions of semi-linear PDE (2.5). It is important to note here that by freezing arguments as above we make (3.11) to be the first order approximation of the solution to the equation (3.10).

Now we can make several observations. Because the prepayment function has a jump, numerical methods, in general, cannot be of the second order approximation unless the discontinuity is specially treated. Additionally, the problem can be magnified in the liability equation because of the non-linear term \( \gamma(L) L \). Namely, the jump in \( \gamma(\cdot) \) could lead to "instability" in \( L \) which, in turn, induce "uncertainty" in \( \gamma(L) \). However, as we will illustrate in the next section, a numerical solution of the liability PDE behaves reasonably well. The reason can be seen if we arrange the terms in the liability PDE (2.5) as follows:

\[
\frac{\partial L}{\partial t} + AL - rL + c - \Phi \gamma(\Phi) = 0,
\]

where \( \Phi = L - P(1 + F) \) and \( \gamma(\cdot) \) is expressed as a function of \( \Phi \). Even though \( \gamma(\Phi) \) is discontinuous as a function of \( \Phi \), the term \( \Phi \gamma(\Phi) \) in the liability PDE is continuous because the jump of \( \gamma(\Phi) \) is located at \( \Phi = 0 \). Moreover, it is Lipschitz continuous. Therefore, the liability PDE of time-continuous version of Stanton's model admits a classical solution. This is not true for the mortgage PDE (2.4). The problem in this case is not "grave" since this PDE is linear. Researchers should be careful if they employ a jump-specification of \( \gamma(\cdot) \) in case of another specification of the option-based prepayment behavior (for example, so called reduced option-based approach in [5]).

Even with a smooth \( \Phi \gamma(\Phi) \), Stanton's computational method is of the first order in time because of two reasons. Firstly, as we pointed out, the second step is of the first order approximation of the original ODE (3.10). Therefore, the method as a whole is of the first order in time. Thus, instead of working with (3.3) we can work with some approximation of equation (3.10) of an appropriate order (or an asymptotic expansion \( (\Delta t \to 0) \) of (3.3)) with the benefit of faster calculations since algebraic operations are performed faster on a computer than evaluations of the exponential functions in (3.3) (recall that the splitting-up method is itself of the second order).

Secondly, the above variant of the splitting-up numerical method is of the first order even if both split equations are solved with the second order. This is because the operators \( B_1 \) and \( B_2 \) in this case are not commutative. To make the procedure be of the second order in time we can switch order of the fractional steps every time interval \([n \Delta t, (n+1) \Delta t]\) (see Marchuk [10]).

In summary, knowing the real PDE standing behind an option-based model, we can be more efficient by using the rich literature on numerical analysis.

4 Numerical Example

Valuation problems of financial engineering usually do not require very precise answers. The measurement errors, model simplicity (with respect to real process being modeled) will inevitably render "unnecessary" precision useless. Therefore, taking into account computational complexity of the mortgage modeling problems (potential high dimensionality, modeling burn-out effect, calibration and valuation of CMO’s) we are interested in getting a "cheap" answer rather then precise one. Therefore, the advantage of a higher order (e.g., Crank-Nicholson) over the first order (e.g., various trees, Euler) numerical schemes is not obvious. The superior performance in the case of small number of time steps is not guaranteed. In this section we show how a second order numerical scheme may dramatically reduce the number of time steps needed to compute the borrower's liability. We use a well-known predictor-corrector rather then a second order version of splitting-up numerical scheme. For extremely small number of steps (the case of interest for us) the former method performs better then the latter.

If the prepayment model is specified through the borrower's liability, the calibration of the model would require repeated computations of that liabilities. Additionally, the heterogeneity of mortgage pools requires the computation of liabilities for "different" borrowers (i.e., solving liability PDEs with different parameters) to account for the burn-out effect. Therefore, the liabilities computation is the
most computationally expensive part of the prepayment modeling process in this set up. Therefore, we illustrate the advantage of knowledge of underlying continuous time model on the non-linear liability PDE (3.5).

The problem in application of a Crank-Nicholson scheme to the non-linear equations (3.5) is that the coefficients cannot be “sampled” between time steps because they depend on yet unknown solution L. Predictor-corrector gives the following solution.

Let $L_n$ be an approximation of $L_{n\Delta t}$ in the notations of section 3.3. We should go backwards in time: given $L_{n+1}$ we have to find $L_n$. In order to do this, we, first, find auxiliary $L_{n+1/2}$ with the help of some first order numerical scheme (predictor); the coefficients of (3.5) can be freezeed at time $t = (n + 1)\Delta t$ for this predictor (since only the first order of convergence is required) and, therefore, are “known” over the interval $[n\Delta t, (n + 1)\Delta t]$. Next, the second order numerical scheme is applied to find $L_n$ given $L_{n+1}$ (corrector); the coefficients of (3.5) are sampled in the middle of the time interval, i.e., the coefficients are computed at $t = (n + 1/2)\Delta t$; in particular $\gamma(L_{(n+1/2)\Delta t})$ is approximated by $\gamma(L_{n+1/2})$. This numerical scheme gives the second order in time approximation of the solution of the quasi-linear PDE (3.5).

In the following example we consider Stanton's specification of the liability equation, i.e., the coefficients $\sigma, a, b, c, \lambda,$ and $\rho$ are taken from Stanton [11]. The transaction cost $F$ is taken to be 34% (estimated average). We computed the liability using the Stanton’s procedure (see Stanton [11]) and the same liability using the predictor-corrector for the liability equation (3.5). Before application of numerical schemes, the equations were transformed according to the domain transformation used in Stanton [11]. The predictor is chosen to be fully implicit Euler scheme and the corrector is chosen to be Crank-Nicholson scheme (the same scheme used by Stanton to compute the “continuous” part of his procedure).

The mortgage price equation (3.6) is a linear PDE and, as we pointed out in the previous section, the computational problem (which is absent in the liability problem) is the jump in a coefficient associated with the intensity function $\gamma(\cdot)$. Since the computation of the price is not a part of the prepayment model calibration, we can accept the first order of convergence (the result of the jump) and compute the price with a “brute force,” i.e., to get satisfactory precision with a finer mesh. Alternatively, since the pricing equation is linear, the problem can be dealt with standard techniques (for example, the domain can be transformed in such a way that the position of the jump is stationary, then we can use a finer mesh in state variables around the jump only) to get the second order of convergence.

Since Stanton’s procedure is of the second order in the interest rate variable $r$, we concentrate merely on the effect of the time variable $t$. So, the number of steps in “space” variable is chosen significantly large (1000). The results are reported for 10, 20, 40 number of time-steps for both methods and 360 for Stanton’s procedure.

As expected, the predictor-corrector method exhibit the second order of convergence, i.e., doubling the number of time-steps reduce the error by about four times (see Table 1). At the same time, the Stanton’s procedure increase precision by about two times, i.e., it is of the first order in time. This pattern for Stanton’s procedure is seemingly broken around the value of 11% for the interest rate. The reason is easy to see on the graph of the relative errors on Figure 1 (the left figure): the errors change sign there and, of course, at some point Stanton’s solution is “exact.” The fact that this solution underprice the liability (and the mortgage) for high and overprice for low value of the interest rate lead to increased error in computation of the delta (if one plans to use the solution to delta-hedge the mortgage security). On Figure 1 (the right graph) the relative errors in delta are shown (computed using the second order center difference). It worth mentioning that the precision in the case of the

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12 The values of $\lambda$ and $\rho$ are taken from Table 1 of Stanton [11].
13 For moderately high dimensions (about 3 or 4 stochastic state variables) we might prefer (Quasi)-Monte Carlo in this case even if the computation of the liabilities with PDE is still more efficient due to the second order of convergence.
14 The number of steps used by Stanton [11].
<table>
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Table 1: Relative errors for various interest rates and numbers of time-steps.

The predictor-corrector can be significantly improved by averaging, while the Stanton’s procedure does not present such an opportunity.

To conclude, we notice that the relative errors of the predictor-corrector with 10 time-steps are less than 0.1% which can be considered as very low for practical applications. Computations performed with 10 space-steps (the interest rate variable) add error of the same magnitude and, therefore, does not change the conclusions.

Figure 1: The graph of relative errors against the interest rate. "PC:n" stands for predictor-corrector with n time-steps, "S:n" stands for Stanton’s procedure with n time-steps.

References


