On the Existence of the Endogenous Mortgage Rate Process*

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Abstract

In this article we provide the first theoretical results on the mortgage rate process for fixed rate mortgages. In the absence of arbitrage the mortgage rate is determined by a distribution of future mortgage payments. At the same time, the mortgage payments depend on the mortgage rate. This presents a complication for mortgage modelers: the circular dependence on the mortgage rates. Generally, this dependence has two reasons: because payments are computed according to the contract rate at the origination, and because the dynamics of the mortgage rates after the origination influences the prepayment behavior and, therefore, the timing when payments stop. If the former reason is taken into account and the latter is ignored (almost all models currently used in industry), then the problem can be reduced to finding only the contract mortgage rate. We give a proof of the convergence of iterations in this case and then we proceed with a more complex case when both dependencies are taken into account. We prove the existence of a solution to this problem in the case when the borrowers prepayments do not depend on higher-than-contract mortgage rates.

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1 Introduction

The major trend on Wall Street to model the prepayment behavior is to use some empirically estimated functions given certain predictors (e.g., [3], [13]). The most influential factor which drives the refinancing (the most “dangerous” from the point of view of investors to conventional fixed rate mortgage securities) is the mortgage rate, more precisely, a quantity of how much the current (available for refinancing) mortgage rate is lower than the contract mortgage rate. In spite of its importance, mortgage rate modeling has attracted little attention from academia and industry in recent years. To the best of this author’s knowledge, the only theoretical literature with prepayment process depending on the mortgage rate process itself consists of his work [5, 6, 7, 8], Pliska [12], and the MOATS model by Citigroup [2]. In other works on the prepayment modeling, some benchmark (e.g., the 10-year Treasury yield or similar predictor) is usually chosen instead of a “true mortgage rate” (e.g., [1], [4], [11]).

It is important to note that any proper prepayment model implies a mortgage rate process (see [5], [6], [10]). If this mortgage rate process coincides with the mortgage rate model used in the prepayment model, then the model is called endogenous. If a prepayment model is not based on the endogenous mortgage rate, then it is flawed, because the prepayment is based on a mortgage rate different from the one being predicted.

The purpose of an endogenous mortgage rate model is to obtain a more reliable than empirical model. A borrower does not look at the Treasury yields or swaptions (in fact, he/she has probably not even heard about them) to make a decision to refinance their mortgages. He/she looks at what is going on with the mortgage rates. In general, he/she has his/her own point of view (which has nothing to do with, e.g., swaptions) on how profitable (if at all) it is to refinance his/her mortgage at a given moment. The art of mortgage prepayment mod-
eling is to quantify it, i.e., to find an adequate refinancing incentive.\(^1\) A “real” model separates borrower’s and market components of a prepayment,\(^2\) i.e., it represents (builds a model of) the prepayment as a function (the borrower’s component, describes/models a human behavior) of the refinancing incentive (the market component, driven by the efficient market). The borrower’s component is “low-frequency”, i.e., we do not expect people’s behavior to change on a short time-scale. The market component, on contrary, is “high-frequency” and might experience a dramatic change in a short period of time. A change in the market would lead to the change in the market component but not in the borrower’s component, thus making the model robust: the change can be handled using standard fixed-income techniques.

The first general endogenous mortgage rate model was developed by Goncharov in [5] and [6], where the endogenous mortgage rate was formulated as a fixed-point of a functional operator. Pliska [12] investigated the problem in a discrete time setting. Goncharov, Okten, and Shah [7] applied a randomized quasi-Monte Carlo method for the mortgage rate computation.

Independently, Citigroup developed its mortgage rate model called MOATS [2]. The model formulation is based on a tree technique: the authors illustrate the model with a help of an example which considers what is happening at each node in a binomial tree. Goncharov and Shah [9] showed that the MOATS model is a variant of the iteration procedure for solution of the endogenous mortgage rate equation proposed in [6, 7] and it produces rates which converge to the endogenous mortgage rates if the “extended interest rate horizon” in the MOATS model increases.

In this paper, we present the first theoretical results about the

\(^1\)The refinancing incentive is assumed to be a function of the current and contract mortgage rates in this paper. The “real” form of this incentive should be the object of separate research which is practically absent at the current time in academic literature.

\(^2\)“Keep meatballs and flies separately” — V.V. Putin.
endogenous mortgage rate process: a proof that such an object does exist. The proof is constructive and the idea is based on a Lebesgue set numerical method. A general description of this method as well as its name are presented to the public public for the first time. The method itself is based on an idea of a recently introduced algorithm proposed to solve the curse of dimensionality in [8]. The advantage of the formulation of the method in this paper (though not the main point of the paper) is that it removes the necessity for the “monotonicity assumption” made in [8]. This assumption might be quite restrictive (e.g., in time-non-homogenous case, the time would be a parameter with respect to which the monotonicity would likely not hold) and is not proven even for the “obvious” case of a one-factor time-homogenous interest rate factor.

2 Model

We assume in our model that the relevant, for our purposes, market factors are described by an integrable (time-homogenous)\textsuperscript{3} Markovian processes \(X_t \in \Omega \subset \mathbb{R}^n, \ t > 0\), where \(\Omega\) is a support of the process \(X_t\). Let us assume that the risk-free instantaneous interest rate \(r_t\) is one of the components of this state vector \(X_t\) and is positive. The other components of the state vector might be home price, unemployment rate, etc. All the objects of our model (which the market “knows”) are adapted to a natural filtration of the \(X_t\) process. In particular, the mortgage rate available for refinancing can be written as \(m_t = m(X_t)\) for some function \(m(x)\) to be defined. We will use notation \(\mathbb{E}_x[\cdot]\) for the conditional expectation \(\mathbb{E}[\cdot | X_0 = x]\).

We consider a level-payment fully amortized \(T\)-year fixed rate mortgage which is originated at time \(t = 0\). In particular, it means that \(m^0\) is the mortgage rate which determines the (constant) mortgage

\textsuperscript{3}Time-inhomogeneous case might be considered in a similar fashion with mainly notational complications.
payments for the whole term of the mortgage contract. The following specification is assumed for the intensity of the prepayment time (see, e.g., [6]; the intensity might be regarded as a prepayment rate for the purpose of this paper):

\[
\gamma_t := \begin{cases} 
\gamma^e(X_t) + \gamma^r(X_t, m^0, m^t), & \text{if } m^0 - m^t > \delta \\
\gamma^e(X_t), & \text{otherwise}
\end{cases}
\]  

(1)

where \( \gamma^r(x, m^0, m^t) \) is the intensity of the interest-rate driven refinancing (based on comparison of the contract \( m^0 \) and current \( m^t \) mortgage rates), \( \gamma^e(x) \) is the intensity of all other prepayments (which are assumed not to be influenced by the mortgage rates), and \( \delta \) is a positive constant which can be interpreted as a minimum number of percentage points which a borrower requires the mortgage rate to fall to cover his/her transaction costs. We assume that the intensity function \( \gamma^r(x, \mu, y) \) is uniformly bounded by \( C_\gamma \) and uniformly continuous with respect to \( \mu \), i.e., \( |\gamma^r(x, \mu, y) - \gamma^r(x, \mu', y)| \leq C(\mu - \mu') \), where \( C(\mu) \to 0 \) when \( \mu \to 0 \). For the intensity defined by (1) we will use \( \gamma_t(m^0, m^t) \) to emphasize the dependency on the mortgage rates.

We assume that for a fixed \( t \in (0, T) \) the distributional derivative \( f_t(\xi|x) \) of \( X_t \) exists and is uniformly (with respect to \( t \) over \( [\varepsilon, T] \) for any \( \varepsilon > 0 \))\(^4\) continuous in \( L_1(\Omega) \) with respect to \( x \), i.e., \( \int_\Omega |f_t(\xi|x) - f_t(\xi|x')|d\xi \leq C_X(||x - x'||, \varepsilon) \), where \( C_X(\sigma, \varepsilon) \to 0 \) for \( \sigma \to 0 \) for any \( \varepsilon > 0 \). This is true for a wide class of processes, such as jump-diffusions or, more generally, processes generated with Levy process.

Note that this assumption excludes processes with, for example, discrete support (e.g., processes which can take only finite number of values).\(^5\)

\(^4\)Uniform continuity is not expected around \( t = 0 \) because typically a probability density function converges to a \( \delta \)-function when \( t \to 0 \).

\(^5\)For such a process in the case of optimal borrower’s behavior it is possible to construct an example which will not allow for the existence of a solution to the endogenous mortgage rate problem (due to Prof. Warren Nichols).
In this paper, we assume that all the borrowers have the “same” prepayment behavior. It is well known not to be true: all human beings are different and this “trivial” fact affects the mortgage modeling in the form of the well known “burn-out” effect. But the extension of the results of this paper to the case of borrower’s heterogeneity presents merely a notational difficulty (see [5, 8]).

Let us define the operator

\[ L[m(\cdot)](\mu, x) := E_x \left[ \int_0^T (\mu - r_t) P(t, \mu) e^{-\int_0^t (\gamma_\theta(\mu, m(X_\theta)) + r_\theta) d\theta} dt \right], \quad (2) \]

where \( P(t, \mu) \) is the outstanding principal at time \( t \) given some contract mortgage rate \( \mu \). Then, as shown in [5, 6, 7], the mortgage rate function \( m(x) \) is a solution of the following nonlinear functional equation:

\[ L[m(\cdot)](m(x), x) = 0 \quad \text{for all } x \in \Omega. \quad (3) \]

All the complications of this problem come from the functional dependence of the operator \( L \) on \( m(\cdot) \) through the intensity \( \gamma_r \). If we treat the prepayment (intensity) model as a “black box” \( \gamma_t \), i.e., if this intensity process is already defined as a function of the state process \( X_t \) but does not explicitly depend upon the mortgage rate, then the problem (3) can be rewritten in the following form

\[ m(x) = A(m(x), x) := \frac{E_x \left[ \int_0^T r_t P(t, m(x)) e^{-\int_0^t (\gamma_\theta + r_\theta) d\theta} du \mid x \right]}{E_x \left[ \int_0^T P(t, m(x)) e^{-\int_0^t (\gamma_\theta + r_\theta) d\theta} du \mid x \right]}, \quad (4) \]

This equation is merely a scalar equation where \( x \) can be regarded as a parameter. The problem is routinely solved using iterative methods by financial institutions involved in the mortgage lending business to know what rate to offer and to know what to expect if the market

\[ ^6 \text{Financial institutions purchase prepayment models as “black boxes” from companies specializing on the prepayment modeling.} \]
changes. In spite of this, the convergence and even the existence and uniqueness of a solution to (4) was not theoretically established up to now. In the following lemma, we establish the existence, uniqueness and continuity of a solution to this equation. At the same time, we show that the iterations do converge.

**Lemma 1.** Equation (4) has a unique continuous solution \( m(x), x \in \Omega \). For a fixed \( x \in \Omega \), the solution \( m(x) \) which can be found with iteration \( m_{n+1} = A(m_n, x) \).

**Proof.** The outstanding principal function for a fully amortized fixed rate level payment mortgage is a smooth and bounded positive function

\[
P(t, \mu) = 1 - e^{-\mu(T-t)} \frac{1 - e^{-\mu T}}{1 - e^{-\mu T}}.
\]

Therefore, the function \( A(\mu, x) \) is continuously differentiable with respect to \( \mu \), \( A(\mu, x) > 0 \), and there is a \( \mu' \) so that \( A(\mu, x) < \mu' \) for a fixed \( x \in \Omega \). Therefore, the existence of a solution follows.

For uniqueness let us note that

\[
\frac{P_{\mu}(t, \mu)}{P(t, \mu)} = \frac{(T-t)e^{-\mu(T-t)}}{1 - e^{-\mu(T-t)}} = \frac{(T-t)}{e^{\mu(T-t)} - 1} < \frac{1}{\mu}.
\]

Therefore

\[
\frac{\partial}{\partial \mu} A(\mu, x) =
\]

and using the first term to bound above and the second term to bound below we obtain

\[
\varepsilon - \frac{A(\mu, x)}{\mu} \leq \frac{\partial}{\partial \mu} A(\mu, x) \leq \frac{A(\mu, x)}{\mu} - \varepsilon
\]

for some \( \varepsilon > 0 \) which is an estimation of dropped terms. It is strictly positive because of the existence of an *a priori* estimate from above.
for a solution (we can take \( P(t, \infty) \) in the numerator and \( P(t, 0) \) in the denominator in (4)). The absolute values of the dropped terms are strictly positive and continuous over a compact interval. Therefore, a nonzero minimum exist for a fixed \( x \). Thus, if \( \mu^* \) is a solution of (4), then

\[
\frac{\partial}{\partial \mu} A(\mu, x) \bigg|_{\mu=\mu^*} < 1.
\]

Therefore, the solution is unique and this defines a function \( m(x) \) for each \( x \in \Omega \).

Continuity follows from the fact that the function \( A(\mu, x) \) is continuous (continuity of the expectations in \( A \) will be proved in Proposition 2 for a more general case). Then the intersection of the continuous surface \( z = A(\mu, x) \) in the space \((\mu, x, z)\) with the continuous hypersurface \( z = \mu \) gives a (continuous) set of all pairs \((\mu, x)\) which solve the equation \( \mu = A(\mu, x) \). But taking into account the (already proven) uniqueness of \( \mu \) for each \( x \in \Omega \), we conclude that this continuous set coincides with the set \( \{(m(x), x)|x \in \Omega\} \). Therefore, the function \( m(x) \) is continuous.

The convergence of the iteration \( m_{n+1} = A(m_n, x) \) follows from the inequality (5). This inequality implies that

\[
\min\left(m, \frac{\mu^*}{m}\right) e^{-\varepsilon|m-m^*|} \leq A(m, x) \leq \max\left(m, \frac{\mu^*}{m}\right) e^{-\varepsilon|m-m^*|}
\]

Assume, without loss of generality that \( m_n < \mu^* \) (otherwise the boundaries of the following intervals should change the order) and consider this inequality for the interval \( I_n = [m_n, \mu^* / m_n] \). It shows that the operator \( A \) maps this interval into the interval

\[
I_{n+1} = [m_n e^{-\varepsilon|m_n-m^*|}, \mu^* / m_n e^{-\varepsilon|m_n-m^*|}],
\]

The sets \( I_n \) converge monotonically to a point \( \mu^* \). Since \( m_n \in I_n \), this proves the convergence \( m_n \to \mu^* \).

Equation (3) is more complex because of functional dependency of the prepayment function on a solution and Lemma 1 does not work.
We address the question of the existence of a solution to equation (3) in the following sections.

3 The Lebesgue set method

A common approach in the literature to solve problem (3) is to perform straightforward iterations like

\[ \mathcal{L}[m_n(\cdot))] (m_{n+1}(x), x) = 0, \]

These iterations (with some variations) were considered in [2, 6, 7, 12]. A common problem of this approach is that to perform one iteration, the operator \( \mathcal{L} \) should be computed for each point on some \( x \)-grid, where each problem’s complexity is of the order of points on this \( x \)-grid. This exponential complexity makes the computation of an endogenous mortgage rate (see [5, 6]) unrealistic for practical applications. The computational price is too high even in the case of the most simple one-factor model (e.g., [9]).

This fundamental complexity problem was solved in [8], where the author proposed to “invert” the problem: instead of finding a mortgage rate for each state factor \( x \), a set of of state factor values is found for each mortgage rate.

The essence of the method lies in an alternative point of view on how a solution is defined. Let us define for every mortgage rate value, \( m \), the “Lebesgue set”

\[ \partial f_m := \{x \in \mathbb{R}^n | \mathcal{L}[f(\cdot)](m, x) = 0 \}. \]

Then, this point of view is established, without formal proof, in the following simple proposition:

**Proposition 1.** The function \( f: \Omega \rightarrow \mathbb{R} \) is a solution of the mortgage rate equation (3) if and only if the Lebesgue hypersurface \( \partial f_m \) is an
Let \( M \) be a set of numbers \( \{m_0, m_1, \ldots, m_n\} \). The following definition of an approximation of a solution takes into account this concept of “being a solution”.

**Definition.** An \( M \)-Lebesgue approximation to a solution of equation (3) is a function such that its Lebesgue hypersurfaces coincide with its \( m \)-level sets for all \( m \in M \).

The idea is somewhat similar to Lebesgue integration (as opposed to Riemann). Therefore, we propose to call a method which finds the \( M \)-Lebesgue approximation a Lebesgue-set method (for details and an explanation of how this method solves the dimensionality problem and removes necessity to iterate, see [8]).

The Lebesgue set idea will serve as a foundation for the proof of the existence of a solution to (3).

### 4 Preliminaries

**Proposition 2.** \( \mathcal{L}(f(\cdot))(\mu, x) \) is a continuous function of \( x \) and \( \mu \) for any measurable \( f(\cdot) \).

**Proof.** Using the triangle inequality and that for any positive \( A \) and \( B \) we have \( |e^{-A} - e^{-B}| \leq |A - B| \), we have the following estimate of a change of \( \mathcal{L} \) with respect to \( \mu \):

\[
|\mathcal{L}(f(\cdot))(\mu, x) - \mathcal{L}(f(\cdot))(\mu', x)| \leq
\]

\[
E_x \int_0^T \left| (\mu - r_t)P(t, \mu)e^{-\int_0^t (\gamma_0(\mu, f(X_\theta)) + r_\theta) d\theta} - (\mu' - r_t)P(t, \mu')e^{-\int_0^t (\gamma_0(\mu', f(X_\theta)) + r_\theta) d\theta} \right| dt \leq
\]
\[ P_0 \mathbf{E}_x \int_0^T |\mu - \mu'| dt + \mathbf{E}_x \int_0^T |\mu - r_t| P(t, \mu) \left| e^{-\int_0^t \gamma_0(\mu, f(X_\theta)) d\theta} - e^{-\int_0^t \gamma_0(\mu', f(X_\theta)) d\theta} \right| dt \leq P_0 T |\mu - \mu'| + P_0 \mathbf{E}_x \int_0^T |\mu - r_t| \int_0^T \left| \gamma_0(\mu, f(X_\theta)) - \gamma_0(\mu', f(X_\theta)) \right| d\theta dt \leq P_0 T |\mu - \mu'| + T C(\mu - \mu') \mathbf{E}_x \int_0 r_t dt. \quad (6) \]

For the estimate of a change of \( L \) due to \( x \), we find it more convenient to use notation \( \mathbf{E} \left[ g(X_\theta^x) \right] \) rather than \( \mathbf{E} \left[ X_\theta^x \right] \), where \( X_\theta^x \) stands for a process conditioned on \( X_0^t = x \). We will need the following representation of \( L \) (see [6]):

\[
L(f(\cdot))(\mu, x) = \mathbf{E} \left[ \int_0^T [c(\mu) + \gamma(\mu, X_\theta^x) P(t, \mu)] \left| e^{-\int_0^t \gamma_0(\mu, f(X_\theta)) + r_\theta d\theta} - e^{-\int_0^t \gamma_0(\mu', f(X_\theta)) + r_\theta d\theta} \right| dt \right] - P_0,
\]

where \( c(\mu) = P_0 \frac{m}{1 - e^{-\rho T}} \) is the borrower’s payment rate. Next, let us choose \( t_\delta \) so that

\[
\mathbf{E} \left[ \int_0^{t_\delta} |\mu - r_t^x| P(t, \mu) e^{-\int_0^t (\gamma_0(\mu, m(X_\theta^x)) + r_\theta^x) d\theta} dt \right] \leq \delta.
\]

Then using the new representation and the triangle inequality, we obtain

\[
|L(f(\cdot))(\mu, x) - L(f(\cdot))(\mu, x')| \leq 2\delta + \mathbf{E} \int_{t_\delta}^T \left| \gamma_0(\mu, f(X_\theta)) - \gamma_0(\mu, f(\mu')) \right| P(t, \mu) e^{-\int_0^t (\gamma_0(\mu, f(X_\theta)) + r_\theta) d\theta} dt + \\
\mathbf{E} \int_{t_\delta}^T [c(\mu) + \gamma(\mu, X_\theta^x) P(t, \mu)] \left| e^{-\int_0^t \gamma_0(\mu, f(X_\theta)) d\theta} - e^{-\int_0^t \gamma_0(\mu', f(X_\theta)) d\theta} \right| dt \leq 2\delta + P_0 \mathbf{E} \int_{t_\delta}^T \left| \gamma_0(\mu, f(X_\theta)) - \gamma_0(\mu, f(\mu')) \right| dt + \\
(c(\mu) + C_T P_0) \mathbf{E} \int_{0}^{T} \left| \gamma_0(\mu, f(X_\theta)) - \gamma_0(\mu, f(X_\theta^x)) \right| d\theta dt \leq 11.
\]
\[ 2\delta + P_0 C_{\gamma} T C_X(||x - x'||, \delta) + \]

\[ (c(\mu) + C_{\gamma} P_0) \mathbb{E} \int_0^T \left( 2C_{\gamma} \frac{\delta}{2C_{\gamma} T} + \int_{2C_{\gamma} T}^t \left| \gamma_{\theta}(\mu, f(X_\theta)) - \gamma_{\theta}(\mu, f(X_\theta')) \right| d\theta \right) dt \leq \]

\[ 2\delta + P_0 C_{\gamma} T C_X(||x - x'||, \delta) + (c(\mu) + C_{\gamma} P_0) \left( \delta + T^2 C_X(||x - x'||, \frac{\delta}{2C_{\gamma} T}) \right) \]

(7)

From the estimates (6) and (7) we conclude that for any \( \varepsilon > 0 \) we can find small \( \delta > 0 \) so that if \( ||\mu - \mu'|| < \delta \) and \( ||x - x'|| < \delta \) then

\[ \left| \mathcal{L}[f(\cdot)](\mu, x) - \mathcal{L}[f(\cdot)](\mu', x') \right| \leq \varepsilon. \]

\[ \square \]

The following proposition looks similar to Lemma 1, but it is essentially different due to the dependency of the intensity \( \gamma_t \) on the unknown \( m(x) \).

**Proposition 3.** Let \( f(x) \) be measurable. Then there exists a measurable function \( m(x) \) such that

\[ \mathcal{L}[f(\cdot)](m(x), x) = 0 \text{ for all } x \in \Omega \]

(8)

Moreover, the infimum of solutions for each fixed \( x \) is a solution as well.

**Proof.** From Proposition 2 we conclude that the function \( z = \mathcal{L}[f(\cdot)](\mu, x) \) represents a continuous surface in the space \( (\mu, x, z) \). From definition (2) of the operator \( \mathcal{L} \) for a fixed \( x \in \Omega \) we have \( \mathcal{L}[f(\cdot)](0, x) < 0 \) and \( \lim_{\mu \to -\infty} \mathcal{L}[f(\cdot)](\mu, x) = \infty \). Therefore, the surface intersects the plane \( z = 0 \) for every \( x \in \Omega \). Let \( I \) be this intersection.

The set \( I \) gives a set of all pairs \( (\mu, x) \) which solve the problem \( \mathcal{L}[f(\cdot)](\mu, x) = 0 \) for all \( x \in \Omega \). Let us define a function \( m(x) := \min\{\mu \mid (\mu, x, 0) \in I\} \), where the minimum exists due to continuity of \( \mathcal{L} \). This function \( m(x) \) is measurable and \( \mathcal{L}[f(\cdot)](m(x), x) = 0 \) for all \( x \in \Omega \) by construction.

\[ \square \]
Proposition 4. Let \( m(x|f(\cdot)) \) be the solution of equation (8) for a given \( f(\cdot) \). Then
\[
m(x|f(\cdot)) \geq \inf_{y \in \Omega} m(y|\infty), \quad \text{for any } x \in \Omega.
\]

Proof. Let \( M := \inf_{x \in \Omega} m(x|f(\cdot)) \). \( M \geq 0 \) because \( m(x|f(\cdot)) \) is bounded from below by zero. Let \( \Omega^* \) be a set of all \( x \in \Omega \) such that \( m(x|f(\cdot)) < M + \delta \). Then equation (8) considered over \( \Omega^* \) does not depend on the \( f(\cdot) \) specification because of the definition of the intensity (1). Thus, \( m(x|f(\cdot)) = m(x|\infty) \) for \( x \in \Omega^* \). Therefore,
\[
m(x|f(\cdot)) \geq M = \inf_{y \in \Omega^*} m(y|f(\cdot)) = \inf_{y \in \Omega^*} m(y|\infty) \geq \inf_{y \in \Omega} m(y|\infty)
\]
for any \( x \in \Omega \). \( \square \)

In the following section we will use the Lebesgue set for a range of mortgage values rather than for one value. That is, for a set of mortgage rates \( M \) we define the Lebesgue set
\[
\partial f_M^L := \bigcup_{m \in M} \partial f_m^L.
\]

5 Existence

Theorem. There exists a function \( m(x) \) which solves the endogenous mortgage rate equation
\[
\mathcal{L}[m(\cdot)](m(x), x) = 0 \quad \text{for all } x \in \Omega.
\]

Proof. We construct a solution by induction. Consider equation
\[
\mathcal{L}[\infty](f_1(x), x) = 0, \quad x \in \Omega. \quad \text{(9)}
\]

\( ^7 \)In what follows, we use the “infinity function \( \infty \)” instead of \( f(\cdot) \) in the operator \( \mathcal{L}[f(\cdot)](\cdot, \cdot) \) whenever we want to exclude a refinancing possibility from the prepayment (all “other rates” are “much higher”). That is, formally, \( m(y|\infty) \) is the solution of equation (8) where the intensity is defined as \( \gamma_t = \gamma^*(X_t) \) (see the intensity \( \gamma_t \) specification (1)).
This is effectively a scalar equation with a parameter $x$. A continuous solution exists due to Lemma 1 because $f_1(x) < \infty$ and, thus, the intensity $\gamma_t$ does not depend on the unknown function $f_1(x)$ (see (1)). Let $m_0 := \inf \{ f_1(x) \}$ and define

$$m_1(x) := \begin{cases} f_1(x) & \text{if } x \in \partial^\infty_{[m_0,m_0+\delta]} \\ \infty & \text{otherwise} \end{cases}$$

Note that since for $x, x_0 \in \partial^\infty_{[m_0,m_0+\delta]}$ the prepayment intensity $\gamma(x, m_1(x_0), m_1(x))$ does not depend on $m_1(x)$, we have $\partial^\infty_{[m_0,m_0+\delta]} = \partial^m_{[m_0,m_0+\delta]}$. Consequently, mortgage rate equation (3) with $m(x) = m_1(x)$ is equivalent to (9) over $\partial^m_{[m_0,m_0+\delta]}$. Therefore, the function $m_1(x)$ solves the endogenous mortgage rate equation over $\partial^m_{[m_0,m_0+\delta]}$. Moreover, any solution $f(x)$ of the equation $L[m_1(\cdot)](f(x), x) = 0$, $x \in \Omega \setminus \partial^m_{[m_0,m_0+\delta]}$ (10) has values larger than $m_0 + \delta$. Indeed, if it were not the case, then for some $x \in \Omega \setminus \partial^m_{[m_0,m_0+\delta]}$ we had $f(x) \leq m_0 + \delta$ and $f(x) \geq m_0$ (the latter from Proposition 4). But that would imply $x \in \partial^m_{[m_0,m_0+\delta]}$.

Let us assume we have found a measurable function $m_n(x)$ such that it solves equation (3) over $\partial^m_{[m_0,m_0+n\delta]}$ and is $\infty$ otherwise. Additionally, assume that any solution $f(x)$ of the equation $L[m_n(\cdot)](f(x), x) = 0$, $x \in \Omega \setminus \partial^m_{[m_0,m_0+n\delta]}$ (11) has values larger than $m_0 + n\delta$.

Consider the equation $L[m_n(\cdot)](f_{n+1}(x), x) = 0$ for $f_{n+1}(x)$. This, again, is effectively a scalar equation with a parameter $x$ because the unknown function $f_{n+1}(x)$ enters the equation as a fixed (to be found) value for a (fixed) parameter $x$. A measurable solution exists according to Proposition 3. Let us chose the minimum solution and define the
function

\[ m_{n+1}(x) := \begin{cases} 
  m_n(x), & \text{if } x \in \partial_{[m_0,m_0+n\delta]}^{m_n} \\
  f_{n+1}(x), & \text{if } x \in \partial_{[m_0+n\delta,m_0+(n+1)\delta]}^{m_n} \\
  \infty, & \text{otherwise} 
\end{cases} \]

By the induction assumption, \( m_n(x) < f_{n+1}(y) \) for any \( x \in \partial_{[m_0,m_0+n\delta]}^{m_n} \) and \( y \in \partial_{[m_0+n\delta,m_0+(n+1)\delta]}^{m_n} \). This shows that the extension \( f_{n+1}(x) \) does not affect the solution \( m_n(x) \) and, therefore, \( m_{n+1}(x) \) solves the mortgage rate equation over \( \partial_{[m_0,m_0+(n+1)\delta]}^{m_n} \). Next, for any \( m \in (m_0 + n\delta, m_0 + (n + 1)\delta) \) we have \( |f_{n+1}(x) - m| < \delta \) and, thus, the intensity \( \gamma(x, m, m_{n+1}(x)) = \gamma(x, m, m_n(x)) \) and

\[ L[m_{n+1}](m, x) = L[m_n](m, x) \]

which implies that \( \partial_{[m_0,m_0+(n+1)\delta]}^{m_{n+1}} = \partial_{[m_0,m_0+(n+1)\delta]}^{m_n} \). Therefore, \( m_{n+1}(x) \) solves the mortgage rate equation over \( \partial_{[m_0,m_0+(n+1)\delta]}^{m_{n+1}} \).

To finish the induction step we need to show that any solution \( f(x) \) of the equation

\[ L[m_{n+1}](f, x) = 0, \quad x \in \Omega \setminus \partial_{[m_0,m_0+(n+1)\delta]}^{m_{n+1}} \]  

(12)

has values larger than \( m_0 + (n + 1)\delta \). Let us assume that there is \( x^* \in \Omega \setminus \partial_{[m_0,m_0+(n+1)\delta]}^{m_{n+1}} \) such that \( f(x^*) \leq m_0 + (n + 1)\delta \). From the induction assumption we have that \( f(x^*) > m_0 + n\delta \). But this implies that \( x^* \in \partial_{[m_0+n\delta,m_0+(n+1)\delta]}^{m_{n+1}} \).

The sequence of functions \( m_n(x) \) is monotonically decreasing and bounded below by the constant \( m_0 \). Therefore the sequence converges pointwise to a measurable function \( m(x) \). To end the proof, we need to show that \( \bigcup_{n=1}^{\infty} \partial_{[m_0,m_0+n\delta]}^{m_n} = \Omega \).

Let \( x^* \in \Omega \). According to Proposition 3, the equation \( L[m](\mu, x^*) = 0 \) has a solution. Such a solution is larger or equal to \( m_0 \) by Proposition 4. Let \( \mu^* \) be the minimum (see Proposition 3) of such possible solutions and let \( n = \lceil (\mu^* - m_0)/\delta \rceil - 1 \). Because \( \mu^* > m_0 + n\delta \), we have \( \gamma(x, \mu^*, m_n(x)) = \gamma(x, \mu^*, m_k(x)) \) for \( k > n \), and this implies \( \gamma(x, \mu^*, m_n(x)) = \gamma(x, \mu^*, m(x)) \). Therefore, \( L[m_n](\mu^*, x^*) = \infty \).
\[ L[m(\cdot)](\mu^*, x^*) = 0. \] At the same time \( \mu^* \leq m_0 + (n + 1)\delta. \) Recall that \( m_{n+1}(x) \) was defined as an extension of \( m_n(x) \) with the function \( f_{n+1}(x) \) and that \( f_{n+1}(x) \) was defined as a minimum of all solutions to \( L[m_n][f_{n+1}(x), x]. \) Since the minimum is unique and \( \mu^* \in (m_0 + n\delta, m_0 + (n + 1)\delta), \) we conclude that \( f_{n+1}(x^*) = \mu^* \) and this implies that \( x^* \in \partial_{[m_0, m_0+n\delta]}^{m_{n+1}}(\cdot) \subset \bigcup_{n=1}^{\infty} \partial_{[m_0, m_0+n\delta]}^{m_n}(\cdot). \] □

References


