On the Existence of the Endogenous Mortgage Rate Process

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Abstract

In this article we provide the first results on the existence of the endogenous mortgage rate process. First, we give with a proof of the convergence of plain iterations in a common for existing prepayment models case: when the prepayment does not depend on the mortgage rate or when some benchmark is used instead of the mortgage rate. Then we proceed with a more complex case when the prepayment is driven (along with other exogenous factors) by a comparison of the current and the contract mortgage rates. Since the true mortgage rate behavior depends on the prepayment specification, the problem leads to a well-known “circular dependence”: the mortgage rate depends on the prepayment, which depends on the mortgage rate, which depends on the prepayment, etc. We prove the existence of a solution to this problem in the case when borrowers prepayment does not depend on higher-than-contract mortgage rates.

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1 Introduction

The major trend on Wall Street to model the prepayment behavior is to use some empirically estimated functions given certain predictors (e.g., [3], [13]). The most influential factor which drives the refinancing (the most “dangerous” from the point of view of investors to conventional fixed rate mortgage securities) is the mortgage rate, more precisely, a quantity of how much the current (available for refinancing) mortgage rate is lower than the contract mortgage rate. In spite of its importance, mortgage rate modeling has attracted little attention from academia and industry in recent years. To the best of this author knowledge, all the literature on the subject consist of his work [5, 6, 7, 8], based on author’s work paper by Pliska [12], and MOATS model by Citigroup [2]. In other works some benchmark (the 10-year Treasury yield or similar predictor) is usually chosen instead of a “true mortgage rate” (e.g., [1], [4], [11]).

It is important to note that any prepayment model implies a mortgage rate process (see [5], [6], [10]). If this mortgage rate process coincides with the mortgage rate model used in the prepayment model, then the model is called endogenous. If a prepayment model is not based on the endogenous mortgage rate, then it is flawed, because the prepayment is based on a mortgage rate different from the one being predicted.

Regrettably, the problem of modeling the mortgage rate is often overseen or viewed as “unimportant”.\footnote{On Wall Street the problem is popular and there were numerous attempts to solve it [private communications with practitioners]. However, only one Wall Street model was open to public [2].} The usual line to dismiss the importance of mortgage modeling research is as follows:\footnote{This is a quote from an actual review of one work on the mortgage rate modeling.}

\textit{We can get around this problem completely by writing prepayment as a function of endogenous factors directly, rather than as a function}
of the mortgage rate (for example, write prepayment as a function of Treasury rates or LIBOR). Now things are simple – just try different values for the current mortgage rate until the model produces a value of par, and were done.

The problem with this logic is that it misses the reason why researchers model in the first place. Shortly put, models exist to “extrapolate” the observations. Models “compress” a studied phenomena to a manageable number of factors. Those factors that are to be historically estimated should ideally be time-homogenous. In such a case an adequate model gives a reliable predictive ability in out-of-sample situations. Now, assume that the prepayment is modeled as a direct function of Treasury yield (as advised above). In this case such a function is called an empirical prepayment model. There is no reason why this function would be time-homogenous. Therefore, the use of historical data results in estimation of “pieces” of snapshots of prepayment behavior at different time. Thus, the reliability of such models is highly questionable. Moreover, every time a market undergoes a structural change (and, in particular, the empirical prepayment function undergoes change), “old” data becomes practically useless.

The purpose of a “real” model is to deal with this kind of problem. A borrower does not look at the Treasury yields or swaptions (in fact, he/she does not even heard about them) to make a decision to refinance their mortgages. He/she looks at what is going on with the mortgage rates. In general, he/she has his/her own point of view (which has nothing to do with, e.g., swaptions) on how profitable (if at all) it is to refinance his/her mortgage at a given moment. The art of mortgage prepayment modeling is to quantify it, i.e., to find an adequate refinancing incentive. A “real” model separates borrower’s

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3 The refinancing incentive is assumed to be a function of the current and contract mortgage rates in this paper. The “real” form of this incentive should be the object of as separate research which is practically absent at the current time in academic literature.
and market components of a prepayment,\footnote{“Keep meatballs and flies separately” — V.V. Putin.} i.e., it represents (builds a model of) the prepayment as a function (the borrower’s component, describes/models a human behavior) of the refinancing incentive (the market component, driven by the efficient market). The borrower’s component is a “low-frequency”, i.e., we do not expect people’s behavior to change on a short time-scale. The market component, on contrary, is “high-frequency” and might experience a dramatic change in a short period of time. A change in the market would lead to the change in the market component but not in the borrower’s component, thus making the model robust: the change can be handled using standard fixed-income techniques.

The first general endogenous mortgage rate model was developed by Goncharov in [5] and [6], where the endogenous mortgage rate was formulated as a fixed-point of a functional operator. Pliska [12] investigated the problem in a discrete time setting. Goncharov, Okten, and Shah [7] applied a randomized quasi-Monte Carlo method for the mortgage rate computation.

Independently, Citigroup developed its mortgage rate model called MOATS [2]. The model formulation is based on a tree technique: the authors illustrate the model with a help of an example which considers what is happening at each node in a binomial tree. Goncharov and Shah [9] showed that the MOATS model is a variant of the iteration procedure for solution of the endogenous mortgage rate equation proposed in [6, 7] and it produces rates which converges to the endogenous mortgage rates if the “extended interest rate horizon” in the MOATS model increases.

In this paper, we present first theoretical results about the endogenous mortgage rate process: a proof that such an object does exist. The proof is constructive and the idea is based on a Lebesgue set numerical method. A general description of this method as well as its name are presented to the public for the first time. The method
itself is based on an idea of recently introduced algorithm proposed to solve the curse of dimensionality in [8]. The advantage of the formulation of the method in this paper (though not the main point of the paper) is that it removes the necessity for “monotonicity assumption” made in [8]. This assumption might be quite restrictive (e.g., in time-non-homogenous case, the time would be a parameter with respect to which the monotonicity would likely not hold) and is not proven even for “obvious” case of one-factor time-homogenous interest rate factor).

2 Model

We assume in our model that the relevant, for our purposes, market factors are described by an integrable (time-homogenous)\(^5\) Markovian processes \(X_t \in \Omega \subset \mathbb{R}^n, \ t > 0\), where \(\Omega\) is a support of the process \(X_t\). Let us assume that the risk-free instantaneous interest rate \(r_t\) is one of the components of this state vector \(X_t\) and is positive. The other components of the state vector might be home price, unemployment rate, etc. All the objects of our model (which the market “knows”) are adapted to a natural filtration of \(X_t\) process. In particular, the mortgage rate available for refinancing can be written as \(m^t = m(X_t)\) for some function \(m(x)\) to be defined. We will use notation \(E_x[\cdot]\) for the conditional expectation \(E[\cdot|X_0 = x]\).

We assume the following specification for the intensity of the pre-payment time (see, e.g., [6]; the intensity might be regarded as a pre-payment rate for the purpose of this paper):

\[
\gamma_t := \begin{cases} 
\gamma^e(X_t) + \gamma^r(X_t, m^0, m^t), & \text{if } m^0 - m^t > \delta \\
\gamma^e(X_t), & \text{otherwise}
\end{cases}
\]

(1)

where \(\gamma^r(x, m^0, m^t)\) is the intensity of the interest-rate driven refinancing (based on comparison of the contract \(m^0\) and current \(m^t\) mortgage

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\(^5\)Time-inhomogeneous case might be considered in a similar fashion with mainly notational complications.
rates), \( \gamma'(x) \) is the intensity of all other prepayments (which are assumed not to be influenced by the mortgage rates), and \( \delta \) is a positive constant which can be interpreted as a minimum number of percentage points which a borrower requires the mortgage rate to fall to cover his/her transaction costs. We assume that the intensity function \( \gamma'(x, \mu, y) \) is uniformly bounded by \( C_\gamma \) and uniformly continuous with respect to \( \mu \), i.e., \( |\gamma'(x, \mu, y) - \gamma'(x, \mu', y)| \leq C(\mu - \mu') \), where \( C(\mu) \to 0 \) when \( \mu \to 0 \). For the intensity defined by (1) we will use \( \gamma_t(m^0, m^t) \) to emphasize the dependency on the mortgage rates.

We assume that for a fixed \( t \in (0, T) \) the distributional derivative \( f_t(\xi|x) \) of \( X_t \) exists and is uniformly (with respect to \( t \) over \( \epsilon, T \)) for any \( \epsilon > 0 \) continuous in \( L_1(\Omega) \) with respect to \( x \), i.e., \( \int_\Omega |f_t(\xi|x) - f_t(\xi|x')|d\xi \leq C_X(||x - x'||, \epsilon) \), where \( C_X(\sigma, \epsilon) \to 0 \) for \( \sigma \to 0 \) for any \( \epsilon > 0 \). This is true for a wide class of processes, such as jump-diffusions or, more generally, processes generated with Levy process.

Note that this assumption excludes processes with, for example, discrete support (e.g., processes which can take only finite number of values).\(^7\)

In this paper, we assume that all the borrowers have the “same” prepayment behavior. It is well known not to be true: all human beings are different and this “trivial” fact affects the mortgage modeling in the form of well known “burn-out” effect. But the extension of the results of this paper to the case of borrower’s heterogeneity presents merely a notational difficulty (see \([5, 8]\)).

Let us define the operator

\[
\mathcal{L}[m(\cdot)|(\mu, x)] := E_x \left[ \int_0^T (\mu - r_t)P(t, \mu)e^{-\int_0^t [\gamma_{\theta}((\mu, m(X_\theta)) + r_\theta)]d\theta} \right],
\]

\( (2) \)

\(^6\)Uniform continuity is not expected around \( t = 0 \) because typically a probability density function converge to a “\( \delta \)-function” when \( t \to 0 \).

\(^7\)For such process in the case of optimal borrower’s behavior it is possible to construct an example which will not allow for the existence of a solution to the endogenous mortgage rate problem (due to Prof. Nichols).
where \( P(t, \mu) \) is the outstanding principal at time \( t \) given some contract mortgage rate \( \mu \). Then, as shown in [5, 6, 7], the mortgage rate function \( m(x) \) is a solution of the following nonlinear functional equation:

\[
\mathcal{L}[m(\cdot)](m(x), x) = 0 \text{ for all } x \in \Omega. \tag{3}
\]

All the complications of this problem come from the functional dependence of the operator \( \mathcal{L} \) on \( m(\cdot) \) through the intensity \( \gamma^r \). If we treat the prepayment (intensity) model as a “black box” \( \gamma_t \), i.e., this process is already defined as a function of the state process \( X_t \), then the problem (3) can be rewritten in the following form

\[
m(x) = A(m(x), x) := \frac{\mathbb{E}_x \left[ \int_0^T r_t P(t, m(x)) e^{-\int_0^t (\gamma^r + r_\theta) d\theta} \, du \right]}{\int_0^T P(t, m(x)) e^{-\int_0^t (\gamma^r + r_\theta) d\theta} \, du} \tag{4}
\]

This equation is merely a scalar equation where \( x \) can be regarded as a parameter. The problem is routinely solved using iterations by financial institutions involved in the mortgage lending business to know what rate to offer and to know what to expect if the market changes.

In the following lemma, we establish the existence, uniqueness and continuity of a solution to this equation. At the same time, we show that the iterations do converge.

**Lemma 1.** Equation (4) has a unique continuous solution \( m(x) \), \( x \in \Omega \). For a fixed \( x \in \Omega \), the solution \( m(x) \) which can be found with iteration \( m_{n+1} = A(m_n, x) \).

**Proof.** The outstanding principal function for a fully amortized fixed rate level payment mortgage is a smooth and bounded positive function

\[
P(t, \mu) = \frac{1 - e^{-\mu(T-t)}}{1 - e^{-\mu T}}.
\]

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8Financial institutions purchase prepayment models as “black boxes” from companies specializing on the prepayment modeling.
Therefore, the function $A(\mu, x)$ is continuously differentiable with respect to $\mu$, $A(\mu, x) > 0$, and there is a $\mu'$ so that $A(\mu, x) < \mu'$ for a fixed $x \in \Omega$. Therefore, the existence of a solution follows.

For uniqueness let us note that
\[
\frac{P_\mu(t, \mu)}{P(t, \mu)} = \frac{(T - t)e^{-\mu(T - t)}}{1 - e^{-\mu(T - t)}} = \frac{(T - t)}{e^{\mu(T - t)} - 1} < \frac{1}{\mu}.
\]
Therefore
\[
\frac{\partial}{\partial \mu} A(\mu, x) = \frac{\mathbb{E}_x \left[ \int_0^T \int \gamma_\theta + r_\theta du \right] - \mathbb{E}_x \left[ \int_0^T \gamma_\theta + r_\theta du \right]}{\mathbb{E}_x \left[ \int_0^T \int P(t, \mu) e^{-\int_0^T (\gamma_\theta + r_\theta) d\theta} du \right]}
\]
and using the first term to bound above and the second term to bound below we obtain
\[
\epsilon - \frac{A(\mu, x)}{\mu} \leq \frac{\partial}{\partial \mu} A(\mu, x) \leq \frac{A(\mu, x)}{\mu} - \epsilon
\]
for some $\epsilon > 0$ which is an estimation of dropped terms. It is strictly positive because of the existence of an a priori estimate from above for a solution (we can take $P(t, \infty)$ in the numerator and $P(t, 0)$ in the denominator in (4)). The absolute values of the dropped terms are strictly positive and continuous over a compact interval. Therefore, a nonzero minimum exist for a fixed $x$. Thus, if $\mu^*$ is a solution of (4), then
\[
\frac{\partial}{\partial \mu} A(\mu, x) \bigg|_{\mu = \mu^*} < 1.
\]
Therefore, the solution is unique and this defines a function $m(x)$ for each $x \in \Omega$.

Continuity follows from the fact that the function $A(\mu, x)$ is continuous (continuity of the expectations in $A$ will be proved in Proposition 2 for a more general case). Then the intersection of the continuous surface $z = A(\mu, x)$ in the space $(\mu, x, z)$ with the continuous hypersurface $z = \mu$ gives a (continuous) set of all pairs $(\mu, x)$ which solve the
equation $\mu = A(\mu, x)$. But taking into account the (already proven) uniqueness of $\mu$ for each $x \in \Omega$, we conclude that this continuous set coincides with the set $\{(m(x), x)|x \in \Omega\}$. Therefore, the function $m(x)$ is continuous.

The convergence of the iteration $m_{n+1} = A(m_n, x)$ follows from the inequality (5). This inequality implies that

$$\min \left( m, \frac{\mu^*}{m} \right) e^{\varepsilon|m-\mu^*|} \leq A(m, x) \leq \max \left( m, \frac{\mu^*}{m} \right) e^{-\varepsilon|m-\mu^*|}$$

Assume, without loss of generality that $m_n < \mu^*$ (otherwise the boundaries of the following intervals should change the order) and consider this inequality for the interval $I_n = [m_n, \mu^*/m_n]$. It shows that the operator $A$ maps this interval into the interval

$$I_{n+1} = [m_ne^{\varepsilon|m_n-\mu^*|}, \mu^*/m_ne^{-\varepsilon|m_n-\mu^*|}]$$

The sets $I_n$ converge monotonically to a point $\mu^*$. Since $m_n \in I_n$, this proves the convergence $m_n \to \mu^*$.  

Equation (3) is more complex because of functional dependency of the prepayment function on a solution and Proposition 4 does not work. We address the question of the existence of (3) in the following sections.

3 The Lebesgue set method

A common approach in the literature to solve problem (3) is to perform straightforward iterations like

$$\mathcal{L}[m_n(\cdot)](m_{n+1}(x), x) = 0,$$

These iterations (with some variations) were considered in [2, 6, 7, 12]. A common problem of this approach is that to perform one iteration, the operator $\mathcal{L}$ should be computed for each point on some $x$-grid,
where each problem’s complexity is of the order of points on this \(x\)-grid. This exponential complexity makes the idea of an endogenous mortgage rate (see [5, 6]) unrealistic for practical applications. The computational price is too high even in the case of the most simple one-factor model (e.g., [9]).

This fundamental complexity problem was solved in [8], where the author proposed to “invert” the problem: instead of finding a mortgage rate for each state factor \(x\), a set of of state factors is found for each mortgage rate.

The essence of the method lies in an alternative point of view on how a solution is defined. Let us define for every mortgage rate value, \(m\), the “Lebesgue set”

\[
\partial^f_m := \{ x \in \mathbb{R}^n | \mathcal{L}[f(\cdot)](m, x) = 0 \}.
\]

Then, this point of view is established in the following simple proposition:

**Proposition 1.** The function \(f(\cdot)\) is a solution of the mortgage rate equation (3) if and only if the Lebesgue hypersurface \(\partial^f_m\) is an \(m\)-level set of \(f(\cdot)\) for all \(m\).

Let \(\mathcal{M}\) be a set of numbers \(\{m_0, m_1, ..., m_n\}\). The following definition of an approximation of a solution takes into account this concept of “being a solution”.

**Definition.** A \(\mathcal{M}\)-Lebesgue approximation to a solution of equation (3) is such a function that its Lebesgue hypersurfaces coincide with its \(m\)-level sets for all \(m \in \mathcal{M}\).

The idea is somewhat similar to Lebesgue integration (as opposed to Riemann). Therefore, we propose to call a method which finds the \(\mathcal{M}\)-Lebesgue approximation a *Lebesgue-set method* (for details and
explanation how this method solves the dimensionality problem and removes necessity to iterate, see [8]).

The Lebesgue set idea will serve as a foundation for the proof of the existence of a solution.

4 Preliminaries

Proposition 2. \( \mathcal{L}[f(\cdot)](\mu, x) \) is a continuous function of \( x \) and \( \mu \) for any measurable \( f(\cdot) \).

Proof. Using the triangle inequality and that for any positive \( A \) and \( B \) we have \( |e^{-A} - e^{-B}| \leq |A - B| \), we have the following estimate of a change of \( \mathcal{L} \) with respect \( \mu \):

\[
\left| \mathcal{L}[f(\cdot)](\mu, x) - \mathcal{L}[f(\cdot)](\mu', x) \right| \leq \int_0^T (\mu - r_t) P(t, \mu) e^{-\int_0^t (\gamma_0(\mu, f(X_t)) + \tau_0) d\theta} - (\mu' - r_t) P(t, \mu') e^{-\int_0^t (\gamma_0(\mu', f(X_t)) + \tau_0) d\theta} \left| dt \right| \leq P_0 E_x \int_0^T |\mu - \mu'| dt + E_x \int_0^T |\mu - r_t| P(t, \mu) e^{-\int_0^t \gamma_0(\mu, f(X_t)) d\theta} - e^{-\int_0^t \gamma_0(\mu', f(X_t)) d\theta} \left| dt \right| \leq P_0 T |\mu - \mu'| + P_0 E_x \int_0^T |\mu - r_t| \left| \gamma_0(\mu, f(X_0)) - \gamma_0(\mu', f(X_0)) \right| d\theta dt \leq P_0 T |\mu - \mu'| + TC(\mu - \mu') E_x \int_0^T r_t dt \tag{6}
\]

For the estimate of a change of \( \mathcal{L} \) due to \( x \), we find it more convenient to use notation \( E[g(X_t^x)] \) rather than \( E_x[X_t^x] \), where \( X_t^x \) stands for a process conditioned on \( X_t^0 = x \). We will need the following representation of \( \mathcal{L} \) (see [6]):

\[
\mathcal{L}[f(\cdot)](\mu, x) = E \left[ \int_0^T [c(\mu) + \gamma(\mu, X_t^x) P(t, \mu)] e^{-\int_0^t (\gamma_0(\mu, m(X_t^x)) + \tau_0) d\theta} dt \right] - P_0,
\]
where \( c(\mu) = P_0 \frac{m}{1 - e^{-\mu T}} \) is the borrower’s payment rate. Next, let us choose \( t_\varepsilon \) so that
\[
E \left[ \int_0^{t_\varepsilon} |\mu - r_t^\varepsilon|^2 P(t, \mu) e^{-\int_0^t (\gamma_0(\mu, m(X^\varepsilon_\theta)) + r_t^\varepsilon) \, d\theta} \, dt \right] \leq \varepsilon.
\]

Then using the new representation and the triangle inequality, we obtain
\[
|\mathcal{L}[f(\cdot)](\mu, x) - \mathcal{L}[f(\cdot)](\mu, x')| \leq 2\varepsilon + P_0 E \int_{t_\varepsilon}^T |\gamma_\theta(\mu, f(X_\theta)) - \gamma_\theta(\mu, f(X^\varepsilon_\theta))| \, dt +
\]
\[
E \int_{t_\varepsilon}^T [c(\mu) + \gamma(\mu, X^\varepsilon_\theta)] P(t, \mu) \left| e^{-\int_0^t (\gamma_\theta(\mu', f(X_\theta)) + r_\theta) \, d\theta} - e^{-\int_0^t \gamma_\theta(\mu', f(X_\theta)) \, d\theta} \right| \, dt \leq 2\varepsilon + P_0 C_\gamma T C_X (||x - x'||, \varepsilon) + (c(\mu) + C_\gamma P_0) E \int_{t_\varepsilon}^T \left( 2C_\gamma \frac{\varepsilon}{2C_\gamma T} + \int_0^t |\gamma_\theta(\mu, f(X_\theta)) - \gamma_\theta(\mu, f(X^\varepsilon_\theta))| \, d\theta \right) \, dt \leq 2\varepsilon + P_0 C_\gamma T C_X (||x - x'||, \varepsilon) + (c(\mu) + C_\gamma P_0) \left( \varepsilon + T^2 C_X (||x - x'||, \frac{\varepsilon}{2C_\gamma T}) \right) (7)
\]

From the estimates (6) and (7) we conclude that for any \( \varepsilon > 0 \) we can find small \( \varepsilon \) and \( \sigma \) so that if \( ||x - x'|| < \sigma \) and \( ||x - x'|| < \sigma \) then
\[
|\mathcal{L}[f(\cdot)](\mu, x) - \mathcal{L}[f(\cdot)](\mu', x')| \leq \varepsilon.
\]
\[
\square
\]
The following proposition looks similar to Lemma 1. But it is essentially different due to the dependency of the intensity $\gamma_t$ on the unknown $m(x)$.

**Proposition 3.** Let $f(x)$ be measurable. Then, there exists a measurable function $m(x)$ such that

$$\mathcal{L}[f(\cdot)](m(x), x) = 0 \text{ for all } x \in \Omega$$

(8)

Moreover, the infimum of solutions for a fixed $x$ is a solution as well.

**Proof.** From Proposition 1 we conclude that the function $z = \mathcal{L}[f(\cdot)](\mu, x)$ represents a continuous surface in the space $(\mu, x, z)$. From definition (2) of the operator $\mathcal{L}$ for a fixed $x \in \Omega$ we have $\mathcal{L}[f(\cdot)](0, x) < 0$ and $\lim_{\mu \to \infty} \mathcal{L}[f(\cdot)](\mu, x) = \infty$. Therefore, the surface intersects the plane $z = 0$ for every $x \in \Omega$. Let $I$ be this intersection.

The set $I$ gives a set of all pairs $(\mu, x)$ which solve the problem $\mathcal{L}[f(\cdot)](\mu, x) = 0$ for all $x \in \Omega$. Let us define a function $m(x) := \min\{\mu | (\mu, x, 0) \in I\}$, where minimum exists due to continuity of $\mathcal{L}$. This function $m(x)$ is measurable and $\mathcal{L}[f(\cdot)](m(x), x) = 0$ for all $x \in \Omega$ by construction. \hfill \Box

**Proposition 4.** Let $m(x|f(\cdot))$ be the solution of equation (8) for a given $f(\cdot)$. Then

$$m(x|f(\cdot)) \geq \inf_{y \in \Omega} m(y|\infty), \text{ for any } x \in \Omega.$$

**Proof.** Let $M := \inf_{x \in \Omega} m(x|f(\cdot))$. $M \geq 0$ because $m(x|f(\cdot))$ is bounded from below by zero. Let $\Omega^*$ be a set of all $x \in \Omega$ such that $m(x|f(\cdot)) < M + \delta$. Then equation (8) considered over $\Omega^*$ does not depend on the $f(\cdot)$ specification because of the definition of the intensity (1). Thus, $m(x|f(\cdot)) = m(x|\infty)$ for $x \in \Omega^*$. Therefore,

$$m(x|f(\cdot)) \geq M = \inf_{y \in \Omega^*} m(y|f(\cdot)) = \inf_{y \in \Omega^*} m(y|\infty) \geq \inf_{y \in \Omega^*} m(y|\infty)$$

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for any \( x \in \Omega \).

In the following section, we will use the Lebesgue set for a range of mortgage values rather than for one value. That is for a set of mortgage rates \( \mathcal{M} \) we define
\[
\partial^f_{\mathcal{M}} := \bigcup_{m \in \mathcal{M}} \partial^f_m.
\]

## 5 Existence

**Theorem.** There exists a function \( m(x) \) which solves the endogenous mortgage rate equation
\[
\mathcal{L}[m(\cdot)](m(x), x) = 0 \text{ for all } x \in \Omega.
\]

**Proof.** We construct a solution by induction. Consider equation
\[
\mathcal{L}[\infty](f_1(x), x) = 0, \quad x \in \Omega.
\]
This is effectively a scalar equation with a parameter \( x \). A continuous solution exists due to Lemma 1 because \( f_1(x) \) is bounded and, thus, the intensity \( \gamma_t \) does not depend on the unknown function \( f_1(x) \) (see (1)).

Let \( m_0 := \inf\{f_1(x)\} \) and define\[m_1(x) := \begin{cases} f_1(x) & \text{if } x \in \partial^{\infty}_{[m_0, m_0+\delta]} \\ \infty & \text{otherwise} \end{cases} \]
Note that since for \( x, x_0 \in \partial^{\infty}_{[m_0, m_0+\delta]} \) the prepayment intensity \( \gamma_t(x, m_1(x), m_1(x)) \) does not depend on \( m_1(x) \), we have \( \partial^{\infty}_{[m_0, m_0+\delta]} = \partial^{m_1(x)}_{[m_0, m_0+\delta]} \). Consequently, mortgage rate equation (3) with \( m(x) = m_1(x) \) is equivalent to (9) over \( \partial^{m_1(x)}_{[m_0, m_0+\delta]} \). Therefore, the function \( m_1(x) \) solves the endogenous mortgage rate equation over \( \partial^{m_1(x)}_{[m_0, m_0+\delta]} \). Moreover, any solution \( f(x) \) of the equation
\[
\mathcal{L}[m_1(\cdot)](f(x), x) = 0, \quad x \in \Omega \setminus \partial^{m_1(x)}_{[m_0, m_0+\delta]} \]

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has values larger than $m_0 + \delta$. Indeed, if it were not the case, then for some \( x \in \Omega \cap \partial_{m_0,m_0+\delta}^{m_1(x)} \) we had \( f(x) \leq m_0 + \delta \) and \( f(x) \geq m_0 \) (the latter from Proposition 4). But that would imply \( x \in \partial_{m_0,m_0+\delta}^{m_1(x)} \).

Let us assume we have found a measurable\(^9\) function \( m_\gamma(x) \) such that it solves equation (3) over \( \partial_{m_\gamma m_0 + n\delta}^{m_\gamma(x)} \) and is \( \infty \) otherwise. Additionally, assume that any solution \( f(x) \) of the equation

\[
\mathcal{L}[m_\gamma(\cdot)](f(x), x) = 0, \quad x \in \Omega \cap \partial_{m_\gamma m_0 + n\delta}^{m_\gamma(x)} \tag{11}
\]

has values larger than \( m_0 + n\delta \).

Consider the equation \( \mathcal{L}[m_\gamma(\cdot)](f_{n+1}(x), x) = 0 \) for \( f_{n+1}(x) \). This, again, is effectively a scalar equation with a parameter \( x \) because the unknown function \( f_{n+1}(x) \) enters the equation as a fixed (to be found) value for a (fixed) parameter \( x \). A measurable solution exists according to Proposition 3. Let us chose the minimum solution and define the function

\[
m_{n+1}(x) := \begin{cases} m_\gamma(x), & \text{if } x \in \partial_{m_\gamma m_0 + n\delta}^{m_\gamma(x)} \\ f_{n+1}(x), & \text{if } x \in \partial_{m_\gamma m_0 + n\delta,m_\gamma + (n+1)\delta}^{m_\gamma(x)} \\ \infty, & \text{otherwise} \end{cases}
\]

By the induction assumption, \( m_n(x) < f_{n+1}(y) \) for any \( x \in \partial_{m_\gamma m_0 + n\delta}^{m_\gamma(x)} \) and \( y \in \partial_{m_\gamma m_\gamma + n\delta,m_\gamma + (n+1)\delta}^{m_\gamma(x)} \). This shows that the extension \( f_{n+1}(x) \) does not affect the solution \( m_n(x) \) and, therefore, \( m_{n+1}(x) \) solves the mortgage rate equation over \( \partial_{m_\gamma m_\gamma + (n+1)\delta}^{m_\gamma(x)} \). Next, for any \( m \in (m_0 + n\delta, m_0 + (n+1)\delta) \) we have \( |f_{n+1}(x) - m| < \delta \) and, thus, the intensity \( \gamma(x,m,m_{n+1}(x)) = \gamma(x,m,m_n(x)) \) and

\[
\mathcal{L}[m_{n+1}(\cdot)](m, x) = \mathcal{L}[m_n(\cdot)](m, x)
\]

which implies that \( \partial_{m_\gamma m_\gamma + (n+1)\delta}^{m_\gamma(x)} = \partial_{m_\gamma m_\gamma + (n+1)\delta}^{m_{n+1}(x)} \). Therefore, \( m_{n+1}(x) \) solves the mortgage rate equation over \( \partial_{m_\gamma m_\gamma + (n+1)\delta}^{m_{n+1}(x)} \).

To finish the induction step we need to show that any solution \( f(x) \) of the equation

\[
\mathcal{L}[m_{n+1}(\cdot)](f(x), x) = 0, \quad x \in \Omega \cap \partial_{m_\gamma m_\gamma + (n+1)\delta}^{m_{n+1}(x)} \tag{12}
\]

\(^9\)We need measurability for the expectations in \( \mathcal{L} \) to be defined.
has values larger than $m_0 + (n + 1)\delta$. Let us assume that there is $x^* \in \Omega \backslash \partial_{(m_0 + n\delta, m_0 + (n + 1)\delta)}^{m_{n+1}}$ such that $f(x^*) \leq m_0 + (n + 1)\delta$. From the induction assumption we have that $f(x^*) > m_0 + n\delta$. But this implies that $x^* \in \partial_{(m_0 + n\delta, m_0 + (n + 1)\delta)}^{m_{n+1}}$.

The sequence of functions $m_n(x)$ is monotonically decreasing and bounded below by the constant $m_0$. Therefore the sequence converges pointwise to a measurable function $m(x)$. To end the proof, we need to show that $\bigcup_{n=1}^{\infty} \partial_{(m_0 + n\delta, m_0 + (n + 1)\delta)}^{m_n} = \Omega$.

Let $x^* \in \Omega$. According to Proposition 3, the equation $L[m(\cdot)](\mu, x^*) = 0$ has a solution. Such a solution is larger or equal to $m_0$ by Proposition 4. Let $\mu^*$ be the minimum (see Proposition 3) of such possible solutions and let $n = \lceil (\mu^* - m_0)/\delta \rceil - 1$. Because $\mu^* > m_0 + n\delta$, we have $\gamma(x, \mu^*, m_n(x)) = \gamma(x, \mu^*, m_k(x))$ for $k > n$, and this implies $\gamma(x, \mu^*, m_n(x)) = \gamma(x, \mu^*, m(x))$. Therefore, $L[m_n(\cdot)](\mu^*, x^*) = L[m(\cdot)](\mu^*, x^*) = 0$. At the same time $\mu^* \leq m_0 + (n + 1)\delta$. Recall that $m_{n+1}(x)$ was defined as an extension of $m_n(x)$ with the function $f_{n+1}(x)$ and that $f_{n+1}(x)$ was defined as a minimum of all solutions to $L[m_n](f_{n+1}(x), x)$. Since the minimum is unique and $\mu^* \in (m_0 + n\delta, m_0 + (n + 1)\delta]$, we conclude that $f_{n+1}(x^*) = \mu^*$ and this implies that $x^* \in \partial_{(m_0 + n\delta, m_0 + (n + 1)\delta)}^{m_{n+1}} \subset \bigcup_{n=1}^{\infty} \partial_{(m_0 + n\delta, m_0 + (n + 1)\delta)}^{m_n}$. □

References


