3.3 The Derivative

For \( y = f(x) \), we define the derivative of \( f \) at \( x \), denoted by \( f'(x) \) to be

\[
\frac{f(x+h) - f(x)}{h} = f'(x)
\]

if the limit exists.

If \( f' \) exists for each \( x \) in \( (a, b) \)

\[ \Rightarrow f \text{ is differentiable over } (a, b) \]

Notes:

1. The process of finding the derivative of \( f \) is called differentiation.

2. \( f' \) gives as:
   
   a) The slope of the T.L \( \frac{dy}{dx} \) of \( y = f(x) \)
   
   b) The instant rate of change

3. The domain of \( f' \) is subset to the domain of \( f \)
Given $f(x) = 8x - 2x^2$

(a) Find $f'$ using the definition
(b) Find the slope at $x = 3$
(c) Find the eqn. of the T.L at $x = 3$

Given $f(x) = 2 - \sqrt{x}$

(a) Find $f'$ using the definition
(b) Find $m$ at $x = 4$
(c) Find the eqn. of the T.L at $x = 4$

Given $f(x) = \frac{2}{x+1}$

Find $f'$ using the definition

Use the following expression and the definition of derivative to find $f'(x)$:

$$f(x+h) - f(x) = -2x^2h + 5xh - 2xh^2 + \frac{5}{2}h^2 - \frac{2}{3}h^3$$
Nonexistence of the derivative

1. If \( f' \) exists at \( x=a \) only if the limit exists at \( x=a \)

2. If the limit DNE at \( x=a \) \( \Rightarrow \) \( f \) is nondifferentiable at \( x=a \) \( \Leftrightarrow \) \( f'(a) \) DNE

3. If \( f \) is not cont. at \( x=a \) \( \Rightarrow \) \( f'(a) \) DNE

\[ \begin{align*}
\lim_{x \to a} f(x) & \neq f(a) & \text{not cont.}
\end{align*} \]

\[ \begin{tikzpicture}
\draw[->] (-1,0) -- (2,0) node[below] {a};
\draw[->] (0,-1) -- (0,2) node[right] {x};
\filldraw (1,1) circle (2pt);
\end{tikzpicture} \]

\[ \begin{align*}
\text{note: if } f \text{ is differentiable at } x=a & \Rightarrow f \text{ must be cont. at } x=a \\
\text{not necessarily true} \quad \leftarrow \quad \Rightarrow
\end{align*} \]

4. If \( f \) has a sharp corner at \( x=a \) \( \Rightarrow \) \( f'(a) \) DNE

\[ \begin{tikzpicture}
\draw[->] (-1,0) -- (2,0) node[below] {a};
\draw[->] (0,-1) -- (0,2) node[right] {x};
\filldraw (1,1) circle (2pt);
\end{tikzpicture} \]

\[ \begin{align*}
\text{no } f' \text{ at } x=a
\end{align*} \]
5. If the graph of $f$ has a vertical tangent at $x = a \Rightarrow f'(a)$ DNE.

V.T. at $x = a$

V.T. at $x = a$

Example: Sketch the graph of $f$ and determine where $f$ is non differentiable.

$$f(x) = \begin{cases} 
2x & x < 2 \\
6 - x & x \geq 2
\end{cases}$$