Correspondence Modules and Their Persistence Diagrams

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Persistence Modules

- In a simple, discrete form (persistence module over $\mathbb{Z}$):

$$\cdots \longrightarrow V_{n-1} \xrightarrow{\phi_{n-1}} V_n \xrightarrow{\phi_n} V_{n+1} \longrightarrow \cdots$$

  Sequence of vector spaces and linear mappings

- Persistence modules $\mathbb{V}$ over $\mathbb{R}$, or more generally, over a poset $(P, \preccurlyeq)$

  Vector spaces V_t, $t \in P$, and linear mappings $\nu_s^t \colon V_s \to V_t$, for any $s \preccurlyeq t$, satisfying:

  (i) $\nu_r^t = \nu_s^t \circ \nu_r^s$, for any $r \preccurlyeq s \preccurlyeq t$;
  (ii) $\nu_t^t = I_{V_t}$, $\forall t \in P$

- In category theory language, $\mathbb{V}$ is a functor

$$\mathbb{V} \colon (P, \preccurlyeq) \to \mathrm{Vec}$$

- Example over $\mathbb{R}$: Homology of a continuously filtered space

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  Sequence of vector spaces with forward or backward mappings

- Can we define zigzags over $\mathbb{R}$? Much less clear
- We introduce *correspondence modules* (c-modules) in which linear mappings are replaced with partial linear relations
- Like persistence modules, these will be defined over any poset $(P, \preccurlyeq)$
- But, first, let us introduce the category CVec

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The Category CVec

- Objects: vector spaces (over a fixed field)
- Morphisms from V to W: linear subspaces of $V \times W$
  The morphisms will be called *correspondences*
- Composition rule: $C_1 : V_1 \to V_2$ and $C_2 : V_2 \to V_3$
  Define $C_2 \circ C_1 : V_1 \to V_3$ as the subspace

$$\begin{aligned} C_2 \circ C_1 = \{ & (v_1, v_3) \in V_1 \times V_3 : \exists v_2 \in V_2 \text{ such that} \\ & (v_1, v_2) \in C_1 \text{ and } (v_2, v_3) \in V_2 \times V_3 \} \end{aligned}$$

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  The graph of $T : V \to W$ gives a correspondence

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- Correspondence modules over $\mathbb{R}$ generalize zigzags and include many other persistence structures

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Level-Set Persistence

- Understand how the topology of the level sets of $f\colon X \to \mathbb{R}$ changes with function values
- Interlevel sets: $X_s^t := f^{-1}([s,t])$, $s \le t$
  Level sets: $X[t] := X_t^t = f^{-1}(t)$, $t \in \mathbb{R}$
- Discrete levels: $\ldots t_{n-1} < t_n < t_{n+1} < \ldots$

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 & & H_*(X_{t_{n-1}}^{t_n}) & & \\
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Continuous Parameter Level-Set Persistence

- For any $s \le t$,

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$$

  consider the correspondence $\nu_s^t = G^*_{\psi_t} \circ G_{\phi_s}$

- Under some additional assumptions, one can show that $\{H_*(X[t]), \nu_s^t\}$ yields a c-module over $\mathbb{R}$
- Carlsson, de Silva, Kališnik, Morozov have studied level-set persistence in the framework of zigzags
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Continuous Parameter Level-Set Persistence

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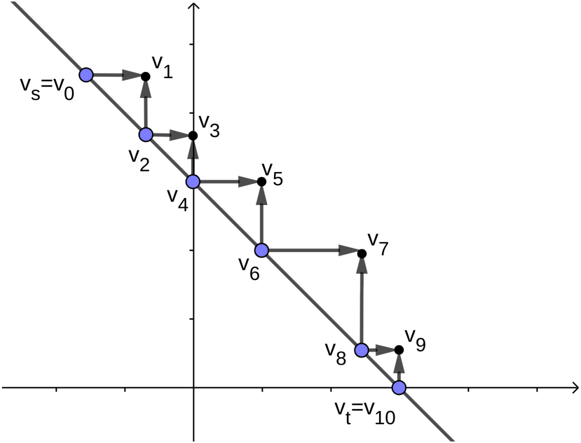

Barcodes Richer in Geometry?

- Sublevel and superlevel filtrations: each gives three infinite bars and a single finite bar in H_0
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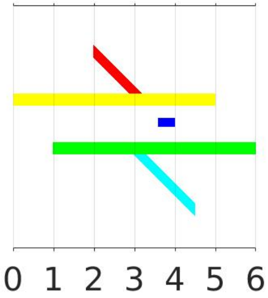

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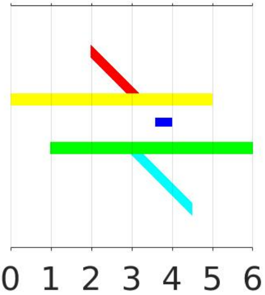

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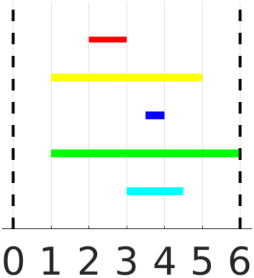

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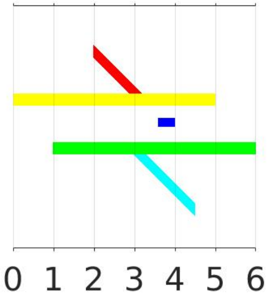

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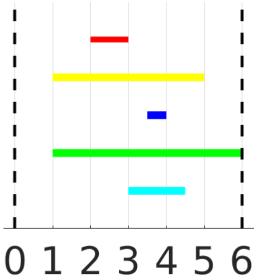

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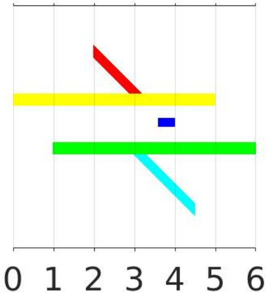

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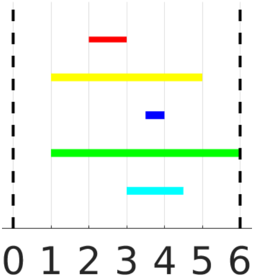

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- As in zigzag persistence, there are up to four types of interval c-modules associated with an interval $I \subseteq \mathbb{R}$
- We denote them $\langle \mathbb{I} \rangle$, $|\mathbb{I}\rangle$, $\langle \mathbb{I}|$, and $|\mathbb{I}|$.
- If $\mathbb{V}$ is any the four, let $V_t = k$, $\forall t \in I$, and $V_t = 0$, otherwise
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- Approach: look at sections of a c-module over $\mathbb{R}$ to obtain a sheaf-like structure
- Sections of a c-module $\mathbb{V}$ over an interval $I \subseteq \mathbb{R}$
  $s = (v_t)_{t \in I}$, where $v_t \in V_t$ and $(v_s, v_t) \in \nu_s^t$, for any $s \leq t$
- $F(I) =$ all sections of $\mathbb{V}$ over the interval I
  This is a vector space
- If $I \subseteq J$, there is a restriction homomorphism $F_I^J \colon F(J) \to F(I)$
- If $I \subseteq J \subseteq K$, then
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- The category Int
  Objects: intervals $I \subseteq \mathbb{R}$
  Morphisms: inclusion of intervals, $I \subseteq J$
- A *persistence pre-sheaf* is a (contravariant) functor

$$F \colon \mathrm{Int}^{\mathrm{op}} \to \mathrm{Vec}$$

- Terminology and notation
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  (ii) (Gluing) If $I = \cup I_\lambda$, $\lambda \in \Lambda$, is a *connected* cover of I by intervals and s_λ are sections over I_λ such that $s_\lambda|_{I_\lambda \cap I_\mu} = s_\mu|_{I_\lambda \cap I_\mu}$, for any $\lambda, \mu \in \Lambda$, then $\exists s \in F(I)$ satisfying $s|_{I_\lambda} = s_\lambda$, for every $\lambda \in \Lambda$.

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  (i) (Locality) If $I = \cup I_\lambda$, $\lambda \in \Lambda$, is a cover of I by intervals and $s \in F(I)$ satisfies $s|_{I_\lambda} = 0$, $\forall \lambda$, then $s = 0$.

  (ii) (Gluing) If $I = \cup I_\lambda$, $\lambda \in \Lambda$, is a *connected* cover of I by intervals and s_λ are sections over I_λ such that $s_\lambda|_{I_\lambda \cap I_\mu} = s_\mu|_{I_\lambda \cap I_\mu}$, for any $\lambda, \mu \in \Lambda$, then $\exists s \in F(I)$ satisfying $s|_{I_\lambda} = s_\lambda$, for every $\lambda \in \Lambda$.

- The cover $[0,2] = [0,1) \cup [1,2]$ is not connected

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- Through p-sheaves, we can analyze the structure of c-modules in the friendlier category Vec-valued functors

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A Decomposition Theorem

Theorem

If F is a tame p-sheaf, then

$$F = \bigoplus_{\text{bounded}} F[\mathbb{I}]^{m_\mathbb{I}} \bigoplus_{\text{r-bounded}} F\langle \mathbb{I}]^{m_\mathbb{I}} \bigoplus_{\ell\text{-bounded}} F[\mathbb{I}\rangle^{m_\mathbb{I}} \bigoplus_{\text{all}} F\langle \mathbb{I}\rangle^{m_\mathbb{I}},$$

where $m_\mathbb{I} \in \mathbb{Z}^+ \cup \{\infty\}$ is the multiplicity of each summand.

- Persistence Diagrams: there are four persistence diagrams associated with a tame *p*-sheaf, one for each type of interval module.
- The persistence diagrams are stable with respect to interleaving

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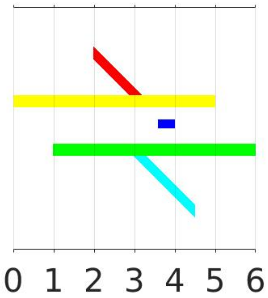
A Fiber-Like Structure



- How to get such barcodes?

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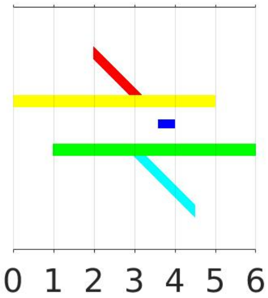

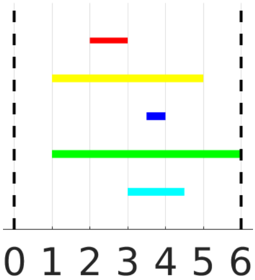

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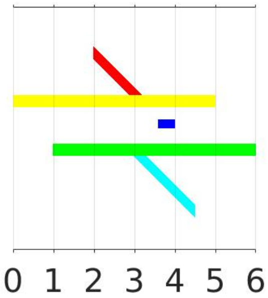

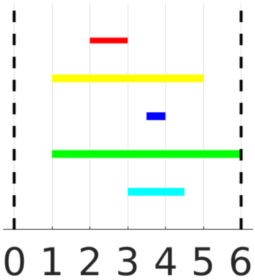

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A (Mayer-Vietoris) Kernel Construction

- $f \colon X \to \mathbb{R}$
  $X^t := f^{-1}(-\infty, t]$ and $X_t := f^{-1}[t, +\infty)$
- Inclusion induced morphisms
  $\iota_t \colon H_*(X^t) \to H_*(X)$ and $\jmath_t \colon H_*(X_t) \to H_*(X)$
- $\ell_t \colon H_*(X^t) \oplus H_*(X_t) \to H_*(X)$, where $\ell_t(a, b) = \iota_t(a) - \jmath_t(b)$
- $K_t := \ker \ell_t \subseteq H_*(X^t) \oplus H_*(X_t)$
- Linear relation $\kappa_s^t \subseteq K_s \times K_t$, $s \leq t$
  $((a_s, b_s), (a_t, b_t)) \in \kappa_s^t \iff \iota_s^t(a_s) = a_t$ and $j_s^t(b_t) = b_s$
- (K_t, κ_s^t) is a c-module (e.g., if X is a compact polyhedron and H_* is Steenrod-Sitnikov)
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