3. Consider Euler's equation

$$x^2y'' + 2xy' + y = 0.$$

(a) Find the general solution valid on the interval $x > 0$.

$\alpha = 2 \quad \beta = 1$

roots $\quad \dfrac{-(\alpha-1)}{2} \pm \dfrac{\sqrt{(\alpha-1)^2-4\beta}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}\,i}{2}$

$y = c_1 x^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\ln x\right)$

(b) Find the limit

$$\lim_{x\to\infty} y(x)$$

where $y(x)$ is any solution, and briefly explain why it doesn't depend on initial conditions.

$\lim_{x\to\infty} y(x) = 0$

since $y = \underbrace{x^{-\frac{1}{2}}}_{\text{goes to zero}} \underbrace{\left(c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 x^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right)}_{\text{bounded}}$

Bonus. Find a homogeneous linear second order differential equation and initial conditions on $y(1)$ and $y'(1)$ that has the particular solution

$$y = x\ln x, \qquad x > 0$$

and no other. repeated roots case

$(\alpha-1)^2 - 4\beta = 0$

$\dfrac{-(\alpha-1)}{2} = 1 \Rightarrow \alpha = -1, \ \beta = 1 \Rightarrow y'' - xy' + y = 0$