Test 1 MAP 2302/3305 – Ordinary Differential Equations

Student's Name: Solutions

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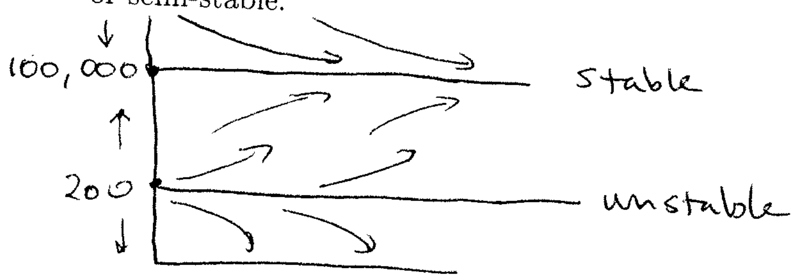
This is an hour and 15 minute in-class test. You may use scrap paper, but please show all work on this sheet.

1. Suppose a population P is modeled by the differential equation:

$$P' = (P - 200)(100,000 - P).$$

(a) Plot the equilibrium solutions and determine which are stable, unstable or semi-stable.



(b) For which initial values will the population stay bounded and not eventually go extinct?

$200 \leq P_0$

2. Solve

$$y'' + 4y' + 5y = 0,$$

with $y(0) = 1$ and $y'(0) = -3$ and sketch a graph of your solution.

Char. eqn. $\quad r^2 + 4r + 5 = 0$

roots $\quad \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

general solution: $\boxed{y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t}$

$$y' = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t$$

$1 = y(0) = c_1$

$-3 = y'(0) = -2c_1 + c_2$

$\Rightarrow c_1 = 1, \; c_2 = -1$

specific solution:

$$\boxed{y = e^{-2t} \cos t - e^{-2t} \sin t}$$

3. Consider the differential equation

$$t^2 y'' - 4ty' + 6y = 0.$$

This has solution $y_1 = t^3$. Find another solution y_2 and verify that it is linearly independent from y_1 by computing the Wronskian.

$y_2 = v y_1$

$w = v'$

$0 = y_1 w' + (2y_1' + p y_1) w$

$= t^3 w' + 2t^2 w$

Solve

$w' + \frac{2}{t} w = 0$

$\frac{w'}{w} = -\frac{2}{t}$

$\int \frac{1}{w}\, dw = -2 \int \frac{1}{t}\, dt$

$w = e^{-2\ln t} = t^{-2}$

$v = -t^{-1}$

$-t^{-1} \cdot t^3 = -t^2$

Conclusion:

$y_2 = t^2$ is another solution

$W(y_1, y_2)$

$$= \begin{vmatrix} t^3 & t^2 \\ 3t^2 & 2t \end{vmatrix}$$

$= 2t^4 - 3t^4$

$= -t^4 \neq 0$ for t

$\therefore y_1, y_2$ are linearly independent.

4. Suppose a salt water tank has 100 gallons of capacity, and a 10% salt water solution flows into the tank at the rate of 10 gallons per minute. Suppose the salt water in the tank is well mixed and exits at the same rate as it enters.

(a) Find a differential equation for the amount of salt in the tank.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= \frac{1}{10} \cdot 10 - \frac{Q}{100} \cdot 10$$

$$= 1 - \frac{Q}{10}$$

$$\boxed{\frac{dQ}{dt} + \frac{Q}{10} = 1}$$

(b) If the tank starts with pure water, give a formula for the amount of salt in the tank at time t.

$$\mu(t) = e^{\int \frac{1}{10} dt} = e^{\frac{1}{10} t}$$

$$Q(0) = 0$$

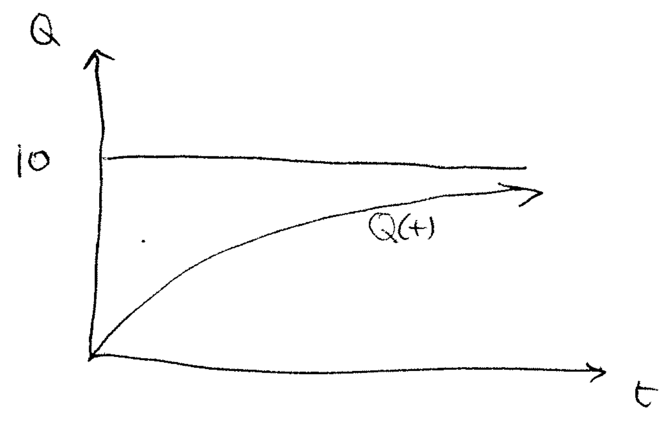
$$e^{\frac{1}{10} t} Q = \int e^{\frac{1}{10} t} dt$$

$$= 10 e^{\frac{1}{10} t} + C$$

$$Q = 10 + C e^{-\frac{1}{10} t}$$

$$\boxed{Q(t) = 10 - 10 e^{-\frac{1}{10} t}}$$

(c) Draw a graph of the amount of salt in the tank, indicating asymptotes.



5. Find the general solution to the non-homogeneous equation

$$y'' + 3y' + 2y = 10e^{3t}.$$

(Hint: the homogeneous solutions are $y_1 = e^{-t}$ and $y_2 = e^{-2t}$.)

General homog solution:

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

$$\boxed{y = \underbrace{c_1 e^{-t} + c_2 e^{-2t}}_{\text{homog. part}} + \underbrace{\tfrac{1}{2} e^{3t}}_{\text{inhomog. part}}}$$

Find non homog. solution:

Try $Y = Ae^{3t}$

$Y' = 3Ae^{3t}$

$Y'' = 9Ae^{3t}$

plug in: $9Ae^{3t} + 9Ae^{3t} + 2Ae^{3t} = 10e^{3t}$

$20A = 10$

$A = \frac{1}{2}$

Bonus. Find a second order, linear, homogenous differential equation satisfied by the function

$$y(t) = e^t(5 - 2t).$$

$(r-1)^2 = r^2 - 2r + 1$

working ~~going~~ backwards

$$\boxed{y'' - 2y' + y = 0}$$