(i) $A \subset \mathrm{Cl}(A)$ and $\mathrm{Cl}(A)$ is a closed set.

(ii) A is closed iff $A = \mathrm{Cl}(A)$.

(iii) $\mathrm{Cl}(A)$ is the smallest closed set containing A.

(iv) If $A \subset B$, then $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$.

(v) $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$.

(vi) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

(vii) $\mathrm{Cl}(A \cap B) \subset \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point $x \in A$ is an *interior point* of A in X provided there is a nbhd N of x with $x \in N \subset A$. The *interior* of A, denoted by $\mathrm{Int}(A)$ or $\mathrm{Int}_X(A)$, is the set of all interior points of A in X.

27. Let $(X, \mathcal{T})$ be a topological space and suppose $A, B \subset X$.

(i) $\mathrm{Int}(A) \subset A$ and $\mathrm{Int}(A)$ is an open set.

(ii) A is open iff $A = \mathrm{Int}(A)$.

(iii) $\mathrm{Int}(A)$ is the largest open set contained in A.

(iv) If $A \subset B$, then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.

(v) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

(vi) $\mathrm{Int}(A \cup B) \supset \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

(vii) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point $x \in X$ is a *boundary point* of A provided each nbhd of x meets both A and $X - A$ (i.e., for every nbhd N if x, $N \cap A \neq \emptyset$ and $N \cap (X - A) \neq \emptyset$). The set of all boundary points of A in X is called the *boundary* of A in X and is denoted by $\mathrm{Bd}(A)$ or $\mathrm{Bd}_X(A)$.

28. Let $(X, \mathcal{T})$ be a topological space and let $A, B \subset X$.

(i) $\mathrm{Bd}(A) = \mathrm{Cl}(A) - \mathrm{Int}(A)$.

(ii) $\mathrm{Bd}(A)$ is closed.

(iii) $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

What can you say about the relationship between $\mathrm{Bd}(A \cap B)$ and $\mathrm{Bd}(A) \cap \mathrm{Bd}(B)$; between $\mathrm{Bd}(A \cup B)$ and $\mathrm{Bd}(A) \cup \mathrm{Bd}(B)$?

Terminology. Whenever it is possible to do so without creating confusion we shall henceforth refer to a "topological space X" or a "space X", meaning a set X with an underlying topology $\mathcal{T}$. A subset A of X is "open" ("closed") provided $A \in \mathcal{T}$ ($(X - A) \in \mathcal{T}$). Likewise, we may refer to a "basis" (or "subbasis") for X or a "basic open set" in X, meaning an underlying subset $\mathcal{B}$(or $\mathcal{C}$) of $\mathcal{T}$ that forms a basis (or subbasis) for $\mathcal{T}$ or one of its members.

Definition. Let X be a topological space. A subset $D \subset X$ is *dense* in X provided $\mathrm{Cl}(D) = X$.

29. The following are equivalent:

(i) D is dense in X.

(ii) If F is closed and $D \subset F$, then $F = X$.

(iii) Each nonempty basic open set meets D.

(iv) Each nonempty open set meets D.

(v) $\text{Int}(X - D) = \emptyset$.

30. Find a countable dense subset of $\mathbb{R}$ (with the standard topology), of $\mathbb{R}^n$.

31. Suppose a subset D of a space X meets every nonempty member of a subbasis. Is D necessarily dense in X?

32. Let Y be a subspace of a topological space X.

(i) $A \subset Y$ is closed in Y iff $A = F \cap Y$ for some closed subset F of X.

(ii) If $A \subset Y$, then $\text{Cl}_Y(A) = Y \cap \text{Cl}_X(A)$.

(iii) If $A \subset Y$, then $Y \cap \text{Int}_X(A) \subset \text{Int}_Y(A)$.

(iv) If $A \subset Y$, then $\text{Bd}_Y(A) \subset Y \cap \text{Bd}_X(A)$.

33. (i) If D is dense in X, is $D \cap Y$ dense in Y?

(ii) Show that equality does not necessarily hold in 32. (iii) and (iv).

Terminology. Let $f: X \to Y$ be a function. There are induced functions $f: \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f(A) = \{f(a): a \in A\}$ for $A \subset X$, and $f^{-1}(B) = \{x \in X: f(x) \in B\}$ for $B \subset Y$.

34. Given a function $f: X \to Y$ and a family $\{B_\gamma\}_{\gamma \in \Gamma}$ of subsets of Y, then

(i) $f^{-1}(\cap_{\gamma \in \Gamma} B_\gamma) = \cap_{\gamma \in \Gamma} f^{-1}(B_\gamma)$.

(ii) $f^{-1}(\cup_{\gamma \in \Gamma} B_\gamma) = \cup_{\gamma \in \Gamma} f^{-1}(B_\gamma)$.

(iii) $f^{-1}(Y - B) = X - f^{-1}(B)$ for $B \subset Y$.

35. Given a function $f: X \to Y$ and a family $\{A_\gamma\}_{\gamma \in \Gamma}$ of subsets of X, establish the appropriate analogs of 34. (i)-(iii).

Definition. Suppose that X and Y are topological spaces. A function $f: X \to Y$ is *continuous at a point* $x \in X$ provided for each nbhd V of $f(x)$ there is a nbhd U of x with $f(U) \subset V$. The function $f: X \to Y$ is *continuous* if it is continuous at each $x \in X$.

36. Given a function $f: X \to Y$, the following are equivalent:

(i) f is continuous.

(ii) $f^{-1}(V)$ is open for each open set V in Y.

(iii) $f^{-1}(C)$ is closed for each closed set C in Y.

Definition. If $f: X \to Y$ and $A \subset X$, the *restriction of f to A* is the function $f|_A: A \to Y$ defined by $f|_A(a) = f(a)$ for all $a \in A$.

37. (i) If $f: X \to Y$ and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous.

(ii) If $f: X \to Y$ is continuous and $A \subset X$ is a subspace, then $f|_A: A \to Y$ is continuous.