Practice Midterm II – with solutions
MAA 4224–Introduction to Analysis, Fall 2011  Eriko Hironaka

The midterm is a 50 minute test. You may use one page of handwritten notes. Here is the rough format.

**Part I: (80 points)** You will be asked to do two problems of the following type.

1. Determine whether the following sets are open and which sets are closed or neither. If the set is not open, find a point in the set that has no ε-neighborhood in the set, if not closed, find a limit point that is not in the set.

   (a) \([0, 1] \cap \mathbb{Q}\) [[not open: none of the points have neighborhoods in the set; not closed: the irrationals are limit points that are not in the set]]

   (b) \(\{\frac{n+1}{n} : n \in \mathbb{N}\}\) [[not open: all the points are isolated, hence they have no neighborhoods in the set; not closed: 1 is a limit point that is not in the set]]

   (c) \(\{1, 2, 3, 4, 5\}\) [[not open: all points are isolated (see above); closed]]

2. Determine which of the following sets are compact. Justify your answers.

   (a) \([0, 1] \cap \mathbb{Q}\) [[not compact since not closed]]

   (b) \(\{\frac{n+1}{n} : n \in \mathbb{N}\}\) [[not compact since not closed]]

   (c) \(\{1, 2, 3, 4, 5\}\) [[compact, since closed and bounded]]

3. Determine which of the following sets are connected. Justify your answers.

   (a) \([0, 1] \cap \mathbb{Q}\) [[not connected; irrational in the set separate the set]]

   (b) \(\{\frac{n+1}{n} : n \in \mathbb{N}\}\) [[not connected: for example, 2 is separated from the rest of the set]]

   (c) \(\{1, 2, 3, 4, 5\}\) [[not connected, any one of the points is separated from the rest]]
4. Find the closures of the following sets. Justify your answers.

(a) \([0, 1] \cap \mathbb{Q} [[The closure is \([0, 1]\), all points in the closed interval are limit points of the set.]]

(b) \(\{\frac{n+1}{n} : n \in \mathbb{N}\} [[\{0\} \cup \{\frac{n+1}{n} : n \in \mathbb{N}\}; 0\) is the only limit point of the set.]]

(c) \(\{1, 2, 3, 4, 5\} [[\text{The set is closed, so the closure is itself}]]

[[In the above problems, the sets could be replaced by other standard ones that have appeared in class. Look through your notes, and become familiar with all the sets that we’ve mentioned, including \(\mathbb{R}, \mathbb{N}, \mathbb{Z}\), all kinds of intervals, and unions of intervals, etc.]]

5. Give an example for each of the following.

1. a countable closed set [[e.g., \(\mathbb{Z}\)]]

2. an open set with a limit point not contained in the open set [[e.g., \((0, 1)\)]]

3. an unbounded closed set [[e.g., \([0, \infty)\), or \(\mathbb{Z}\)]]

4. a connected bounded set [[e.g., \([0, 1]\)]]

5. an isolated point of a closed set: [[e.g., 1 in \(\{1\}\)]]

6. a limit point of an open set [[e.g., 1 in \((0, 2)\)]]

[[Think of others...]]

**Part II: Do any one of the following. (20 pts) (You will be given a choice out of two.) [[These were done in class]]

1 Prove using the definition of limit point that if \(x_n \in X\) for all \(n \in \mathbb{N}\), and the sequence \((x_n)\) converges to \(x \in \mathbb{R}\), then \(x \in \overline{X}\).

2. Prove that if \(X\) is a closed set and \(I\) is the collection of isolated points of \(X\), then \(X \setminus I\) is closed.
3.

\[
    f(x) = \begin{cases} 
        x & \text{if } x \in \mathbb{Q} \\
        0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}
    \end{cases}
\]

1. Sketch the graph.

2. Prove that \( f(x) \) is continuous at \( x = 0 \).

3. Prove that \( f(x) \) is discontinuous at \( c = 100 \).

4. Prove that \( f(x) = 3x + 17 \) is continuous at \( c = 5 \).