The dynamics of mapping classes on surfaces

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1 Introduction to mapping classes and the minimum dilatation problem

In this section, we define mapping classes on surfaces, and briefly state some theorems and conjectures that motivate the content of this course. We refer to [FLP], [CB], [Thu], [FM] and references therein for more thorough expositions on properties of mapping classes.

Let S be a compact oriented surface of genus g with n boundary components. Here, by surface, we mean a topological manifold of dimension two. Two homeomorphisms $f, g : S \to S$ are *isotopic* if there is a continuous map

$$\mathfrak{h} : S \times I \to S$$

so that

(i) the maps

$$\mathfrak{h}_t : S \to S$$

defined by $\mathfrak{h}_t(x) = \mathfrak{h}(x,t)$ are homeomorphisms,

(ii) $f = \mathfrak{h}_0$ and $g = \mathfrak{h}_1$.

We will write $f \sim g$ if f is isotopic to g. Then $\sim$ is an equivalence relation. The map $\mathfrak{h}$ is called an *isotopy* of f and g.

A *mapping class* is an equivalence class of orientation preserving homeomorphisms $f : S \to S$ considered up to isotopic equivalence.

Exercise 1 *If $f_i \sim g_i$, for $i = 1, 2$, then $f_1 \circ f_2 \sim g_1 \circ g_2$.*

It follows that composition is well-defined for mapping classes.

Corollary 1.1 *The set of mapping classes forms a group with group operation given by composition.*

The group of mapping classes is denoted by $\mathrm{MCG}(S)$.

Nielsen-Thurston Classification.

Assume that S has negative Euler characteristic. Under the Nielsen-Thurston classification [Thu], the elements of $\mathrm{MCG}(S)$ partition into three types: periodic, reducible, and pseudo-Anosov. One way to describe these types of mapping classes is to look at their actions on simple closed curves on S.

A *closed curve* g on S is a continuous map

$$\gamma : [0,1] \to S,$$

where either $\gamma(0) = \gamma(1)$ or $\gamma(0), \gamma(1) \in \partial S$. In the latter case, we sometimes say g is *relatively* closed. A closed curve is *simple* if it never crosses itself. Two simple closed curves γ_0 and γ_1 are *isotopic* if there is an isotopy

$$\mathfrak{g} : S \times [0,1] \to S$$

so that $\mathfrak{g}_0$ is the identity map, and $\mathfrak{g}_1 \circ \gamma_0 = \gamma_1$. A simple closed curve is *trivial* if it is isotopic to the trivial loop $\gamma_{\mathrm{triv}}(t) = p$ for some $p \in S$ or to a boundary component. A simple closed curve that is not trivial is called *essential.* A *multi-curve* is a finite union of simple closed curves whose images in S are mutually disjoint.

Exercise 2 *Let $f_1, f_2 : S \to S$ are isotopic homeomorphisms, and γ, τ isotopic simple closed curves on S, then $f_1 \circ \gamma$ is isotopic to $f_2 \circ \tau$. Conversely, if $f : S \to S$ is a homeomorphism and γ, τ are simple closed curves on S such that $f \circ \gamma$ is isotopic to τ, then there is a homeomorphism $f_1 : S \to S$ isotopic to f such that $f_1 \circ \gamma = \tau$.*

Corollary 1.2 *Given a simple closed curve γ and $\phi \in MCG(S)$, the isotopy class $[\phi(\gamma)]$ is determined by the isotopy class $[\gamma]$.*

A mapping class $\phi \in \mathrm{MCG}(S)$ is *periodic* if ϕ^n contains the identity map on S. Nielsen showed that ϕ is periodic if and only if ϕ contains a homeomorphism f such that f^n is the identity on S. In particular, for any simple closed curve γ on S, $\phi^n(\gamma)$ is isotopic to γ.

A mapping class $\phi \in \mathrm{MCG}(S)$ is *reducible* if ϕ is not periodic, and for some simple closed curve γ on S, $\phi^n(\gamma)$ and γ are in the same isotopy class. For such a ϕ and γ, there is a representative f of ϕ so that $f^n \circ \gamma = \gamma$. Let $\gamma_i = f^i \circ \gamma$ for $i = 0, \dots, n-1$. This defines a multi-curve $\tau = \gamma_0 \cup \cdots \cup \gamma_{n-1}$ fixed by f. Thus, if ϕ is reducible, then ϕ has a representative f that fixes a multi-curve $\tau \subset S$ and permutes the complementary components of τ in S.

There are several ways to define pseudo-Anosov mapping classes, and we will give equivalent definitions in later lectures. In all definitions, the essential idea is that $\phi \in \mathrm{MCG}(S)$ is pseudo-Anosov, if the action of ϕ on S is exponentially expanding on simple closed curves. To make this definition rigorous, we need to explain what we mean by the length of a simple closed curve. To do this, requires additional data.

Recall that a Riemannian metric ω on S is an assignment of inner products on the tangent spaces of points on S that vary continuously as one moves around S. A Riemannian metric allows one to compute angles and lengths on the surface. The length $\ell_\omega(\gamma)$ of γ with respect to a fixed Riemannian metric is the infimum of lengths of elements in the isotopy class of γ. If S has negative topological Euler characteristic, then for each simple closed curve γ on S, $\ell_\omega(\gamma) = 0$ if and only if γ is homotopic to a trivial loop or to a boundary curve.

A sequence of integers a_n has exponential growth rate $\lambda > 1$ if

$$\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = \lambda.$$

Assume that S has negative Euler characteristic. A ϕ is *pseudo-Anosov*[1] if for some Riemannian metric ω on X $\ell_\omega(\phi^n(\gamma))$ grows exponentially for all closed curves γ on X. If ϕ is pseudo-Anosov, then $\ell_\omega(\phi^n(\gamma))$ grows exponentially for all Riemannian metrics on X and the growth rate $\lambda(\phi) = \lambda$ is also independent of ω and γ (see, for example, [FM] Theorem 14.23). The growth rate λ is called the *dilatation* of ϕ.

Using Nielsen's work, Thurston proved the following classification theorem for mapping classes.

Theorem 1.3 (Nielsen,Thurston [Thu]) *Let S be an orientable surface with negative Euler characteristic. Then every mapping class in $MCG(S)$ is either periodic, reducible, or pseudo-Anosov.*

Two motivating open problems guide the contents of the series of lectures.

Problem 1 (Minimum dilatation problem) *Study the dilatations of pseudo-Anosov mapping classes of compact surfaces as a collection of algebraic integers. How does the minimum dilatation behave as a function of g and n?*

Problem 2 *Describe families of pseudo-Anosov mapping classes that admit asymptotically small dilatation.*

The dilatations of a pseudo-Anosov mapping class are algebraic integers of degree bounded in terms of the genus g and number of boundary components n. This means that for fixed g and n, the minimum exists and is larger than one. It is known further that the dilatations belong to a special

[1] The more standard definition of pseudo-Anosov is given in terms of the existence of stable and unstable invariant foliations [Thu].

class of algebraic integers called reciprocal Perron numbers. It is an open problem whether all reciprocal Perron numbers occur as dilatations of mapping classes.

So far, the minimum dilatation is known only for small (g, n), namely $(1, 1), (2, 0)$ and $(0, n)$ for $n = 4, \ldots, 9$ [KLS], [HS], [CH], [LT].

Exercise: Find the minimum dilatation for $(g, n) = (1, 1)$ *and* $(0, 4)$.

It is known that the minimum dilatation $\delta_{g,n}$ behaves asymptotically like

$$\log(\delta_{g,n}) \asymp \frac{1}{\chi_{g,n}}$$

where $\chi_{g,n} = |\chi(S_{g,n})$ is the absolute value of the Euler characteristic, where (g, n) varies along lines $n = 0$ [Pen], $g = 0$ [HK]. $g = 1$ [Tsa], and lines through the origin in the (g, n)-plane [Val]. For lines defined by fixed $g \geq 2$, however, the asymptotic behavior is different, and we have [Tsa]

$$\log(\delta_{g,n}) \asymp \frac{\log \chi_{g,n}}{\chi_{g,n}}.$$

A family of mapping classes $\mathcal{F}$ *admits asymptotically small dilatation* if for every $\phi : S \to S$ in $\mathcal{F}$

$$\lambda(\phi)^{|\chi(S)|} \leq C$$

for some fixed C.

In the following sections, we will look at two general constructions of pseudo-Anosov mapping classes with small dilatation. The first construction uses graphs, and properties of Artin and Coxeter groups. The second involves what we will call a deformation of a mapping class to higher dimensions. That is, we start with a mapping class on a small surface, and successively extend the mapping class to higher genus surfaces in which the original surface is embedded, first by extending the mapping class by the identity map, and then composing by a periodic map.

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