The dynamics of mapping classes on surfaces

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10 Coxeter mapping classes and graphs with tails.

In this lecture, we will study examples of mapping classes coming from Coxeter graphs (see, for example, [Lab], [LP], [Mat], [Hir2]). In particular, we will investigate the dilatation of mapping classes associated to graphs with tails. We will show how such mapping classes can lie on fibered faces.

A *(simply-laced) Coxeter graph* is a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Assume that each edge connects two distinct vertices, and there is at most one edge between each pair of vertices. We will also consider *mixed-sign Coxeter graphs*, which are Coxeter graphs with labels $+$ or $-$ on each of the vertices. A graph Γ is ordered if there is an ordering of the vertices:

$$\mathcal{V} = \{v_1, \dots, v_k\}.$$

A 1-dimensional subcomplex A of S *fills* S if every connected component of $S \setminus A$ is a disk or a boundary parallel annulus.

Let S be an oriented surface and $\gamma_1, \dots, \gamma_n$ be a configuration of simple closed curves. The pair $(S, \{\gamma_1, \dots, \gamma_n\})$ is called a *realization* of an ordered graph Γ if

(i) $i_{\mathrm{alg}}(\gamma_i, \gamma_j) = 1$ if and only if $i > j$, and

(ii) the union of curves $\gamma_1 \cup \cdots \cup \gamma_n$ fills S.

Let δ_i be right Dehn twist centered at γ_i, $i = 1, \dots, n$. Let

$$\phi_\Gamma : S \to S$$

be given by

$$\phi_\Gamma = \delta_n \circ \cdots \circ \delta_1.$$

We call ϕ_Γ, the *Coxeter mapping class* associated to Γ.

Coxeter systems. We give here a brief review of the definition of Coxeter systems associated to a simply-laced Coxeter graph Γ. For further references, see [Bour] [Hum] [A'C].

Let A_Γ be the adjacency matrix for Γ. That is,

$$A_\Gamma = [a_{i,j}],$$

where

$$a_{i,j} = \begin{cases} 1 & \text{if there is an edge from } v_i \text{ to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Let $B_\Gamma = 2I - A$. This defines a symmetric bilinear form on $V_\Gamma = \mathbb{R}^n$, where we identify standard basis vectors $e_1, \dots, e_n$ with the vertices of Γ.

To each vertex v_i we associate a reflection $s_i : V_\Gamma \to V_\Gamma$ through the hyperplane transverse to e_i with respect to B_Γ. In equations

$$\begin{aligned} s_i : \mathbb{R}^n &\to \mathbb{R}^n \\ v_j &\mapsto v_j - 2\mathrm{Proj}_{e_i} e_j \\ &= v_i - B(e_i, e_j) v_i. \end{aligned}$$

The *Coxeter element* associated to Γ is the product

$$\omega_\Gamma = s_1 \cdots s_n.$$

Mixed-sign Coxeter graphs We define a *mixed-sign Coxeter graph* Γ to be a Coxeter graph Γ where vertices are labeled with $+1$ or -1. Let $B_\Gamma = I_s - A_\Gamma$, where I_s is the diagonal matrix with entries 1 or -1 depending on the sign of the ith vertex of Γ. Let W_Γ be the relfection system defined by B_Γ as in the classical case.

For any realization $(S, \{\gamma_1, \dots, \gamma_n\})$ of the underlying graph of Γ, let

$$\phi_\Gamma = d_{\gamma_1}^{s_1} \circ \cdots \circ d_{\gamma_n}^{s_n}.$$

where s_i is the label of v_i.

Relation between $\boldsymbol{\omega_\Gamma}$ and $\boldsymbol{\phi_\Gamma}$. Although ω_Γ preserves a bilinear form, while the action $(\phi_\Gamma)_* : H_1(S;\mathbb{R}) \to H_1(S;\mathbb{R})$ preserves a symplectic form defined by the intersection form in $H_1(S;\mathbb{R})$, there two transformations are conjugate.

Given a square matrix T over $\mathbb{R}$, let $\lambda(T)$ be the spectral radius of T.

Theorem 1 *The homological action $(\phi_\Gamma)_*$ of ϕ_Γ on $H_1(S;\mathbb{R})$ is conjugate to minus the Coxeter element ω_Γ of Γ. That is, there is an isomorphism*

$$\beta : V_\Gamma \simeq H_1(S;\mathbb{R})$$

so that the diagram

$$\begin{tikzcd} V_\Gamma \arrow[r, "(\phi_\Gamma)_*"] \arrow[d, "\beta"'] & V_\Gamma \arrow[d, "\beta"] \\ H_1(S;\mathbb{R}) \arrow[r, "-\omega_\Gamma"] & H_1(S;\mathbb{R}) \end{tikzcd}$$

commutes.

Theorem 1 is useful in the context of dilatations of pseudo-Anosov mapping classes in light of the following.

Theorem 2 ([Ryk]) *The homological dilatation is a lower bound for geometric dilatation. That is, if ϕ is a pseudo-Anosov mapping class then*

$$\lambda(\phi_*) \leq \lambda(\phi),$$

with equality if and only if the invariant foliations $(\mathcal{F}^\pm, \nu^\pm)$ are orientable.

Furthermore, Theorem 1 can be used to prove the following.

Theorem 3 *If Γ is a connected mixed-sign Coxeter graph and $\lambda(\omega_\Gamma) > 1$, then ϕ_Γ is pseudo-Anosov, and $\lambda(\phi_\Gamma) \geq \lambda(\omega_\Gamma)$.*

Proof. If Γ is connected, then ω_Γ has no invariant subspaces. Therefore, ϕ_Γ is irreducible. By the Thurston-Nielsen classification, the only possibilities are ϕ_Γ is periodic or pseudo-Ansoov. Since periodic maps have homological dilatation equal to 1, ϕ_Γ cannot be periodic. The rest follows from Theorem 2. □

Example: A_n-graph. A *spherical Coxeter system* is one where the graph $W(\Gamma)$ generated by the reflections s_i is finite. This is equivalent to the requirement that B_Γ defines a positive or negative definite form. In particular, all the elements of $W(\Gamma)$ are periodic. The spherical (classical) Coxeter systems are completely classified. One sequence of spherical Coxeter systems is defined by the graphs A_n shown in Figure 1. The Coxeter group $W(A_n)$ is isomorphic to the symmetric group on $n+1$ elements, and the Coxeter element ω_{A_n} corresponds to an $n+1$-cycle.

Figure 1: A_n graph with $n = 8$ vertices

The A_n graph has realization shown in Figure 2.

A bipartite ordering of the vertices of a graph $\{v_1, \ldots, v_n\}$ is one where for some $1 < s < n$, the subgraphs of Γ generated by $\{v_1, \ldots, v_s\}$, and by $\{v_{s+1}, \ldots, v_n\}$ have no edges.

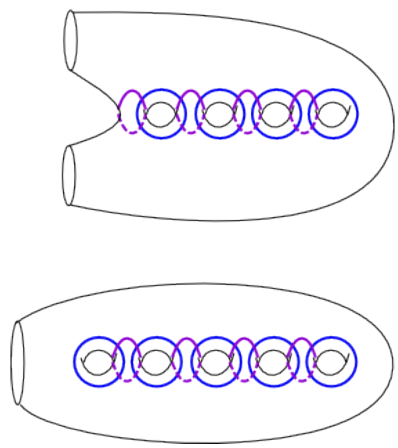

Figure 2: Realization of A_n graph, $n = 6$ and $n = 7$

Proposition 4 *Let A_n have either the sequential or the bipartite order. For n odd, $(\phi_{A_n})^{n+1}$ is a Dehn twist on a curve parallel to the boundary of S_{A_n}. For n even, $(\phi_{A_n})^{n+1}$ is a composition of Dehn twists on the boundary components composed with the hyperelliptiic involution.*

The surface S_{A_n} and mapping class ϕ_{A_n} can also be pictured as a periodic map on a closed surface (see Figure 3). This picture of S_{A_n} is due to Veech, and emphasizes a flat structure preserved by the action of ϕ_{A_n}. The surface is a union of two symmetric polygons attached along parallel edges. (Only parallel edges from the two different polygons are identified.) The mapping class ϕ_{A_n} is a roatation by one-click composed composed with the involution given by interchanging the polygons.

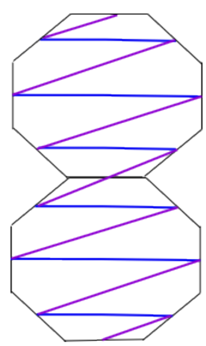

Figure 3: Realization of A_n graph on a closed surface: Veech construction $n = 7$.

Murasugi Sum A *Murasugi sum* of two mapping classes (S_1, ϕ_1) and (S_2, ϕ_2) is defined as follows. Let K be a polygon with sides $s_1, \ldots, s_k$ and let $\iota_i : K \hookrightarrow S_i$ be embeddings so that

1. $\iota_1(s_i) \subset \partial S_1$, $\iota_2(s_i) \subset \mathrm{Int}(S_2))$ for i even, and
2. $\iota_1(s_i) \subset \mathrm{Int}(S_1)$, $\iota_2(s_i) \subset \partial S_2$.

Then the *Murasugi sum* of (S_1, ϕ_1) and (S_2, ϕ_2) with respect to (K, ι_1, ι_2), is (S, ϕ), where S is obtained by attaching S_1 and S_2 along the image of K by ι_1 and ι_2, and ϕ is the composition of mapping classes $\tilde{\phi}_2 \circ \tilde{\phi}_1$, where $\tilde{\phi}_i$ is the extension of ϕ_i by the identity.

For $n = 1$, (S_{A_1}, ϕ_{A_1}) is an annulus with a positive Dehn twist along its core curve. The mapping torus for (S_{A_1}, ϕ_{A_1}) is the complement of a neighborhood of the Hopf link in the three sphere S^3.

Let (S, ϕ) be any mapping class, and let α be any path on S whose endpoints lie on ∂S. Then by thickening α one obtains the embedded image of a square of the required type to do a Murasugi sum. Take a path β on the annulus S_{A_1} that has endpoints on each of the boundary components of S_{A_1}. Then β is determined up to isotopy. The *Hopf plumbing* of (S, ϕ) determined by α is the Murasugi sum of (S, ϕ) and (S_{A_1}, ϕ_{A_1}) along the images of the square determined by α and β.

Coxeter links Let $D \subset S^3$ be an embedded disk in S^3. A *chord diagram* is an ordered union of distinct line segments $\ell_1, \dots, \ell_k$ on D. Then there is an associated mapping class (S, ϕ) obtained by Hopf plumbing along $\ell_1, \dots, \ell_k$. The ℓ_i determine core loops $\alpha_1, \dots, \alpha_k$ on S that generate $H_1(S; \mathbb{Z})$.

Theorem 5 ([Hir1], cf. Murasugi [Mur]) *The mapping class (S, ϕ) corresponding to an ordered chord diagram has mapping torus homeomorphic to $S^3 \setminus L$, where L is the boundary of the surface in S^3 obtained by attaching positively twisted bands to D along thickenings of $\ell_1, \dots, \ell_k$. If the ordering of the lines preserves the order of their slopes, then $\alpha_1, \dots, \alpha_k$ is a realization of the incidence graph of $\ell_1, \dots, \ell_k$, and (S, ϕ) is the corresponding Coxeter mapping class.*

Figure 4 gives a chord diagram together with its dual graph.

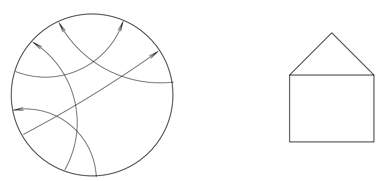

Figure 4: Chord diagram and dual graph.

Figure 5 gives a picture of a Coxeter link.

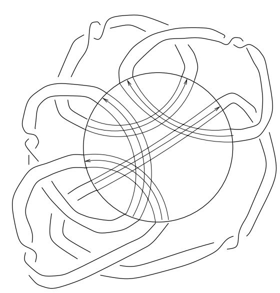

Figure 5: Coxeter link associated to a chord diagram.

Figure 6 shows the effect of iterated Hopf plumbing on fibered links.

Graphs with tails. From the above discussion, it follows that joining two graphs along a vertex amounts to doing a Murasugi sum along squares. Figure 7 gives an example of a chord diagrams associated to a join.

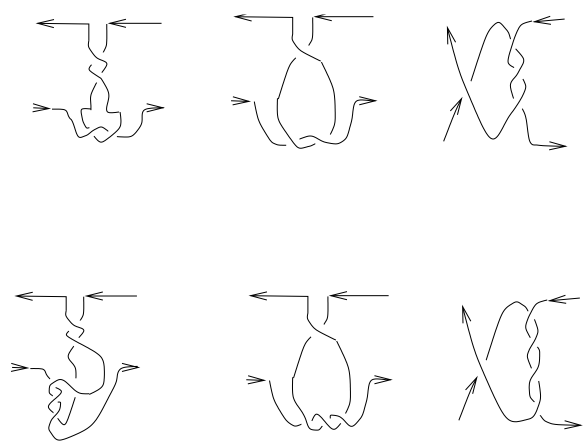

Figure 6: Two iterations of Hopf plumbing.

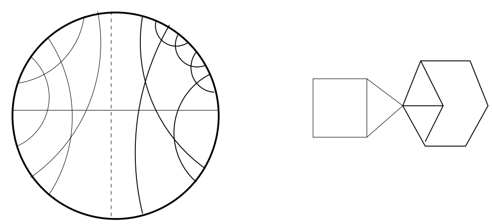

Figure 7: The join of two graphs at a vertex, and the corresponding chord diagarm.

In particular, we can attach A_n to any graph at a vertex (Figure 8).

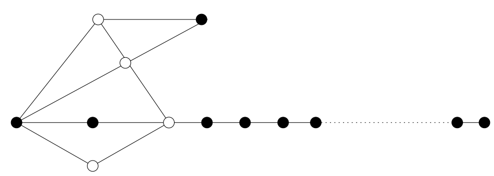

Figure 8: A graph with a tail.

Homological dilatation of Coxeter mapping classes with tails. Let M be the mapping torus for (S, ϕ) and let (S_n, ϕ_n) be the Murasugi sum of (S, ϕ) with (S_{A_n}, ϕ_{A_n}). We say that (S_n, ϕ_n) is obtained from (S, ϕ) by iterated Hopf plumbing.

Theorem 6 ([Hir1]) *Let (S_n, ϕ_n) be obtained by iterated Hopf plumbing from a single mapping class. Then the characteristic polynomial for the homological action of ϕ_n is given by*

$$\Delta_n(x) = x^n f(x) + f^*(x),$$

for some polynomial $f(x)$.

Corollary 7 *The sequence $\lambda_{hom}(\phi_n)$ converges, and the limit is greater than 1.*

Coxeter mapping classes with tails and fibered faces. Mapping classes with tails naturally lie on fibered faces of a single 3-manifold.

Lemma 8 *Let M be the mapping torus for (S, ϕ), and let (S_1, ϕ_1) be the monodromy of a fibered link. Let M_1 be the mapping torus for the Murasugi sum of (S, ϕ) and (S_1, ϕ_1). Then the Dehn fillings of M and M_1 induced by the fibrations are homeomorphic manifolds.*

Proof. The filling of M_1 is obtained from the filling of M by removing a ball, and then gluing it back in by a homeomorphism that fixes the boundary. □

Theorem 9 *Let (S_n, ϕ_n) be obtained by iterated Hopf plumbing. Then for large n, (S_n, ϕ_n) has a singular orbit $\mathcal{O}_n$ of order 2 corresponding to the center of the polygons making up S_{A_n}. Let M_n be the mapping torus for (S_n, ϕ_n), and let M_n' be M_n minus the suspension of $\mathcal{O}_n$. Then there is a smooth point on S with orbit $\mathcal{O}$ under ϕ so that $M \setminus \mathcal{O}$ is homeomorphic to $M_n \setminus \mathcal{O}_n$.*

Corollary 10 *The sequence (S_n, ϕ_n) is made up of convergent subsequences on fibered faces of a single 3-manifold, and all accumulation points lie on the boundary of a fibered face.*

Proof. Since the homological dilatation is a lower bound for geometric dilatation, the normalized dilatations of the mapping classes must go to infinity. □

Adding more than one tail. For the graph given in Figure 9, we have the following (cf. [Tsa])

Theorem 11 (H-Kin [HK])

$$\log \lambda(\phi_{n,n}) \asymp \frac{\log(n)}{n}.$$

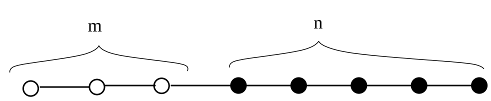

Figure 9: Brinkman example.

It follows that these mapping classes also accumulate toward the boundary of fibered faces of two 3-manifolds (depending on the relative parity of m, n).

Question 12 *Do mapping classes obtained by adding less than or equal to two tails always accumulate on the boundary?*

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