The dynamics of mapping classes on surfaces

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2 Train tracks, transition matrices

In this lecture, we define train tracks on surfaces, and show how they can be used to find the dilatation of a mapping class. Some references are [BH] and [PH].

A *retraction* of a space X onto a subspace $A \subset X$ is a continuous map $r : X \to A$ so that the restriction of r to A is the identity map. The map r is a *deformation retract* if there is a continuous map

$$\mathfrak{h} : X \times I \to X$$

such that $h(x, 0) = x$ and $h(x, 1) = r(x)$, in other words, a deformation retract is a retraction that is homotopic to the identity map on X.

A *graph* Γ is a finite union of points $\mathcal{V}$ called *vertices*, and line segments $\mathcal{E}$ called *edges* so that the end points of each edge lie in $\mathcal{V}$. Then Γ has a natural topological structure. Assigning positive length values to each edge gives Γ a metric structure. An *edge-path* on Γ is a path $[0, 1] \to \Gamma$ that is the composition of a finite sequence of edges on Γ that are connected end-to-end, so that no two consecutive edges are the same, but traversed in the opposite direction. If each edge has a length value, then the length of an edge-path γ is the sum of the lengths of edges (counting multiplicity) that make up γ.

Let S be a surface (compact, oriented) and let $\Gamma \subset S$ be an embedded graph. We call Γ a *spine* of S if S has a deformation retract $r : S \to \Gamma$.

Given a closed curve γ on S, and a spine Γ of S, $r(\gamma)$ defines an isotopy class of closed edge-paths on Γ. We define the *length* of γ to be the minimum length of elements in the isotopy type of $r(\gamma)$.

A *half-edge* is an edge together with a choice of endpoint (ϵ, v). The *degree* of a vertex v is the number of distinct half-edges (ϵ, v) with endpoint v. If Γ lies on a surface S, then for every vertex v of Γ, the orientation on S induces a cyclic ordering of the half-edges meeting at v. A graph with this cyclic structure at vertices is called a *fat graph*.

Exercise 1 *Given a fat graph, there is a corresponding oriented surface S such that Γ embeds in S as a spine, and the fat graph structure on Γ is the same as the one induced by S.*

A *train track* τ is an embedded trivalent graph with the following extra structure. At each vertex v, let (ϵ_i, v) be the half-edges with endpoint v, $i = 1, 2, 3$. A *smoothing* of Γ at v is a choice of tangent line L at v so that each (ϵ_i, v) is tangent to L at v, and (ϵ_i, v) meets (ϵ_j, v) either *in a cusp* or *smoothly* as in Figure 1.

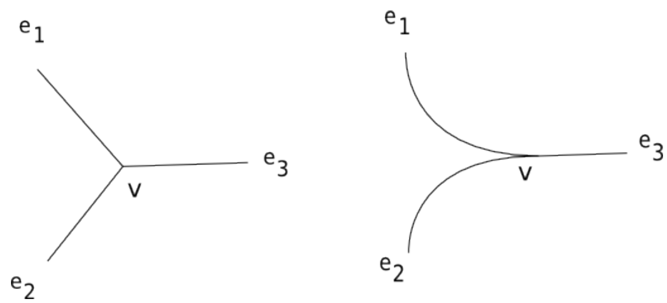

Figure 1: The edges e_1, and e_2 meet in a cusp, while e_1, e_3 and e_2, e_3 meet smoothly at v

An *edge-path* on a train track is an edge-path on the underlying graph of τ such that each pair of consecutive edges meet smoothly at a vertex. An edge-path is *closed* if its initial and endpoints are the same. By the definition of edge-paths on train tracks it is not possible for an edge-path to back track.

Let $\tau \subset S$ be a train track, with a deformation retract $r : S \to \tau$. A closed curve γ on S is *carried by* τ if $r(\tau)$ is isotopic to a closed edge-path on τ.

A *train track map* is the isotopy class of a continuous map $f : \tau \to \tau$ that permutes the vertices of τ and sends each edge in τ to an edge-path with no backtracks. We do not require that the isotopies preserve the image of vertices. Let V be the $\mathbb{R}$-span of the edges of τ. Then train track maps induce linear maps $F : V \to V$. If we fix an ordering on the edges of τ, then F has a matrix representation, which is called the *transition matrix* for the train track map f. Closed curves γ carried by τ are in one-to-one correspondence with vectors $v_\gamma \in V$ with non-negative integer values. By the definitions, for each closed curve γ on S carried by τ,

$$v_{\phi(\gamma)} = F v_\gamma. \qquad (1)$$

Given a mapping class $\phi : S \to S$, a train track τ is *compatible* with ϕ if there is a deformation retraction $r : S \to \tau$ so that $r \circ \phi|_\tau$ is a train track map.

Lemma 0.1 *If τ is compatible with ϕ, then for each essential closed curve γ on S carried by (τ, r), $r(\phi(\gamma))$ has no backtracks.*

Theorem 0.2 *A mapping class $\phi : S \to S$ is pseudo-Anosov if and only if there is a train track τ in S that is compatible on S, so that for all essential closed curves γ on S, we have*

(i) for large enough n, $\phi^n(\gamma)$ is carried on τ;

(ii) for large enough n, $v_{\phi^n(\gamma)}$ has strictly positive entries; and

(iii) $|\phi^n(\gamma)|$ has growth rate $\lambda > 1$.

Here the length function $|\gamma|$ on curves γ carried by τ is defined in terms of any fixed choice of positive length values for the edges of τ, and λ is independent of this choice and equals the dilatation of ϕ.

Example 1. Consider again the example ϕ from Lecture 1 known as the "smallest" hyperbolic mapping class. As you iterate the map, you can see that the edges map to edge-paths that are carried on the train track. The transition matrix is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

This implies that the lengths of essential curves on S are growing, with growth rate equal to the largest root of

$$x^2 - 3x + 1.$$

Thus, $\lambda = \frac{3+\sqrt{5}}{2}$.

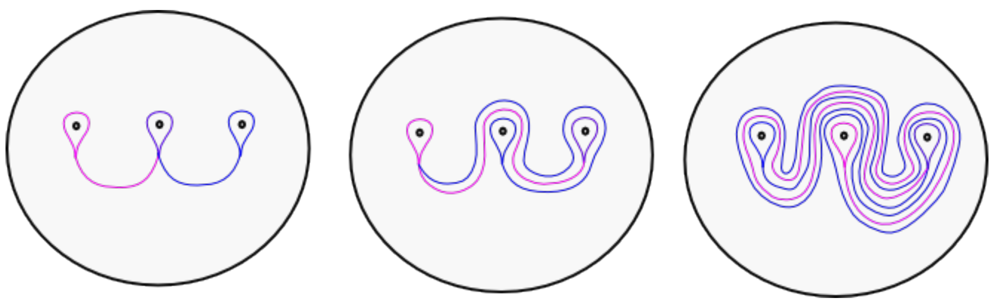

Figure 2: The "smallest" hyperbolic mapping class

Example 2. A *right Dehn twist* along an essential simple closed curve γ on a surface S is the map obtained by "cutting" S along γ, and then regluing after a full right hand twist.

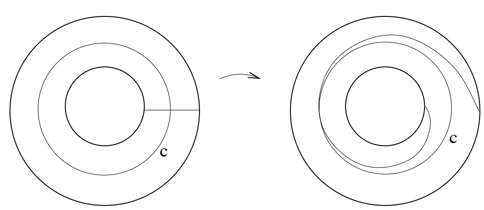

Figure 3: Right Dehn twist

A *left Dehn twist* turns the other direction. Consider the mapping class $\widetilde{\phi}$ on the torus with one boundary component $(g, n) = (1, 1)$ given by doing a right Dehn twist along the meridian composed with a left Dehn twist along a longitude. The transition matrix for this example is the same transition matrix as in Example 1. There is an easy explanation using covering space. One can think of the punctured torus as the double cover of a disk branched along 3 points. Then $\widetilde{\phi}$ is the "lift" of the mapping class ϕ from Example 1. That is, there is commutative diagram

$$\begin{array}{ccc} S_{1,1} & \xrightarrow{\widetilde{\phi}} & S_{1,1} \\ \downarrow & & \downarrow \\ S_{0,4} & \xrightarrow{\phi} & S_{0,4} \end{array}$$

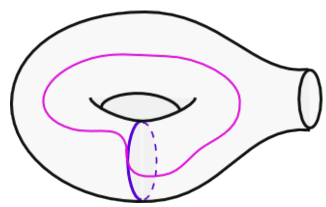

Figure 4: Torus example

References

[BH] M. Bestvina and M. Handel. Train-tracks for surface homeomorphisms. *Topology* **34** (1994), 1909–140.

[CH] J. Cho and J. Ham. The minimal dilatation of a genus two surface. *Experiment. Math.* **17** (2008), 257–267.

[HS] J-Y Ham and W. T. Song. The minimum dilatation of pseudo-Anosov 5-braids. *Experimental Mathematics* **16** (2007), 167,180.

[PH] R. Penner and J. Harer. *Combinatorics of Train Tracks*, volume 125 of *Ann. of Math. Stud.* Princeton University Press, 1991.