The dynamics of mapping classes on surfaces

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3 Train track automata

In this lecture, we define train track automata. Good resources are [KLS], [HS].

In Lecture 2, we defined a *train track* τ as a trivalent fat graph, with a smoothing at each vertex. A train track determines a surface S and a deformation retract $r : S \to \tau$. A mapping class $\phi : S \to S$ is compatible with τ if $r \circ \phi|_\tau$ is a train track map up to isotopy.

We now define an operation on the set of all train tracks. Let τ be a train track. Given an edge e on τ there are two half-edges of e associated to the two possible orientations on e. Given an orientation $\vec{e}$ on e, denote by $-\vec{e}$ the oppositely oriented edge. Thus,

$$-(-\vec{e}) = \vec{e}.$$

We identify $\vec{e}$ and $-\vec{e}$ with their corresponding half-edges. Let $v(\vec{e})$ and $v(-\vec{e})$ be the corresponding endpoints of e. Since τ is a fat graph, any pair of half-edges $\vec{e}$ and $\vec{f}$ meeting at v have an associated ordering. We will write

$$\vec{e} \succ \vec{f}$$

if $\vec{f}$ comes directly after $\vec{e}$ in the counter-clockwise ordering of the half-edges at v.

Folding map. Let $\vec{e}$ and $\vec{f}$ be two half-edges meeting at a cusp at $v = v(\vec{e}) = v(\vec{f})$, where $\vec{e} \succ \vec{f}$. Let $w = v(-\vec{f})$, and let $\vec{g}$ and $\vec{h}$ be the half-edges meeting $-\vec{f}$ at w so that the counter-clockwise ordering at w is given by

$$-\vec{f} \succ \vec{g} \succ \vec{h}.$$

Let τ' be the train track obtained from τ by

(i) removing the vertex v and adding a vertex v' in the midpoint of g,

(ii) replacing the half-edges $\vec{e}, \vec{k}$, and the full-edge f with the half-edges $\vec{e'}, \vec{g'}, \vec{f'}$ and full-edge g'.

This is shown in Figure 1. We denote the folding map by

$$\tau \xrightarrow{[e\to f]} \tau'$$

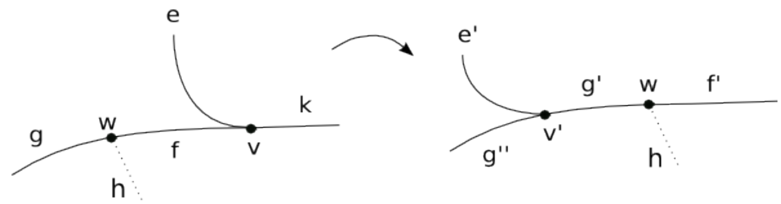

Figure 1: Folding map.

Remark: Folding is the inverse of an operation on train tracks known as *sliding* if $\vec{h}$ meets $\vec{g}$ in a cusp, and known as *splitting* if $\vec{h}$ meets $\vec{g}$ smoothly.

The following can be verified from the definition.

Lemma 1 *The number of edges and number of vertices of a train track is preserved under folding.*

As in Lecture 2, let V_τ be the real vector space spanned by the edges of τ, and let v_e be the basis vector associated to the edge e, called an *edge vector*. Then V_τ contains the space of edge-paths on τ. A train track map between train tracks τ and τ' is any linear map $T : V_\tau \to V_{\tau'}$ that sends edges to edge-paths.

We think of the vectors in the dual space $V_\tau^* = \mathrm{Hom}(V_\tau, \mathbb{R})$ as labels on the edges of τ. If τ and τ' are related by a folding, then there is a corresponding linear map, which we also call the *folding map*,

$$T^* : V_{\tau'}^* \to V_\tau^*$$

such that

$$\begin{array}{ccc} v_{e'}^* & \to & v_e^* \\ v_{f'}^* & \to & v_e^* + v_f^* \\ v_{g'}^* & \to & v_e^* + v_g^* \\ v_{g''}^* & \to & v_g^* \end{array}$$

and the rest of the edge vectors stay are preserved.

Remark: We are often interested in the subspace of V_τ^* consisting of vectors whose coefficients satisfy switching conditions, since these generalize the notion of closed curves on a surface.

The train track map $T : V_\tau \to V_{\tau'}$ associated to the folding is the dual to T^*.

A *train track automaton* is a graph with vertices associated to train tracks, and directed edges connecting a vertex τ to a vertex τ' if τ' is obtained from τ by a folding. Each directed edge from τ to τ' has a corresponding folding map $T_{\tau,\tau'}^* : V_{\tau'} \to V_\tau$ and dual train track map $T_{\tau\tau'}$ defined as above.

Theorem 2 (Ko, Los, Song [KLS]) *Let $T : V_\tau \to V_\tau$ be any train track map. Then there is a directed cycle on a train-track automaton*

$$\tau = \tau_0 \to \tau_1 \to \cdots \to \tau_N = \tau$$

so that T is the dual to the composition of folding maps associated to the arrows in the cycle, and T is given by

$$T = T_{\tau_{N-1},\tau_N} \circ \cdots \circ T_{\tau_0,\tau_1} = (T^*_{\tau_0,\tau_1} \circ T^*_{\tau_1,\tau_2} \circ \cdots \circ T^*_{\tau_{N-1},\tau_N})^*.$$

For convenience, we simplify train track automata by defining an equivalence on train tracks, and combining certain folding operations.

Generalized vertices. If $\vec{e}_0, \ldots \vec{e}_{k-1}$ is a collection of oriented edges on τ so that for $i = 0, \ldots, k-1$ we have

(i) $\vec{e}_i$ and $-\vec{e}_{i+1(\mathrm{mod}\ k)}$ have the same endpoint;

(ii) $\vec{e}_i \succ -\vec{e}_{i+1(\mathrm{mod}\ k)}$

we call the union $e_0, \ldots, e_{k-1}$ a *generalized vertex.* An edge is called an *infinitessimal edge* if it is part of a generalized vertex. The figure on the right of Figure 2 is a generalized vertex. Since $k \geq 3$, we call $k-2$ the *order* of the vertex.

A *simplified train track automaton* is one where the vector space V_τ are generated by the vertices that are not infinitessimal, and two train tracks related by a folding as in Figure 2 are identified. Thus, for simplified train tracks we allow some actual vertices to have degree more than 3.

Figure 2: A generalized vertex and identification of train tracks

Example 1: The simplest hyperbolic braid example is obtained by a cycle of length 2 in a simplified train track automaton with a single vertex and two edges. The labels on the vertices indicate the defining information for the transition matrix associated to the corresponding folding map. See Figure 3.

Example 2: The example in Figure 4 represents a cycle in a simplified train track automaton, where the single arrows represent folding maps, and the double arrow is an identification. Thus, the figure represents a cycle of order 3 on a train track automaton.

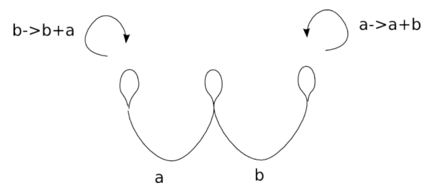

Figure 3: Automaton for simplest hyperbolic braid example

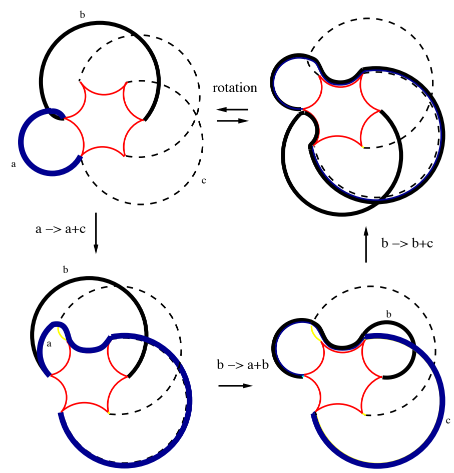

Figure 4: Genus 2 example with minimum dilatation

References

[HS] J-Y Ham and W. T. Song. The minimum dilatation of pseudo-Anosov 5-braids. *Experimental Mathematics* **16** (2007), 167,180.

[KLS] K.H. Ko, J.E. Los, and W.T. Song. Entropies of Braids. *J. of Knot Theory and its Ramifications* **11** (2002), 647–666.