# The dynamics of mapping classes on surfaces

Eriko Hironaka

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## 4 Perron-Frobenius matrices

In this lecture we explicitly relate the study of digraphs to the problem of understanding the algebraic integers that arise as dilatations of pseudo-Anosov mapping classes. To do this, we discuss the definition and properties of Perron-Frobenius matrices (see, for example, [Mey]). Perron-Frobenius theory is relevant to us, since dilatations are the spectral radius (maximum norm of eigenvalues) of such matrices.

**Perron-Frobenius matrices and pseudo-Anosov mapping classes.** A vector $v \in \mathbb{R}^n$ or a matrix $F$ is *positive* (respectively, *non-negative*), written $v > 0$ (resp., $v \geq 0$) or $F > 0$ (resp., $F \geq 0$), if all its entries are positive (resp., non-negative). A matrix $F$ is called *Perron-Frobenius* if

(i) $F \geq 0$, and

(ii) for some $k > 0$, $F^k > 0$.

Since $F$ is a non-negative matrix, $F^k > 0$ implies $F^{k'} > 0$ for all $n' \geq n$.

The *spectral radius* of a matrix $F$ is defined as follows:

$$\mathrm{Spec}(F) = \max\{|\mu| \ : \ \mu \text{ is an eigenvalue of } F\}.$$

**Proposition 1** *Let $\lambda$ be the dilatation of a pseudo-Anosov mapping class. Then it is the spectral radius of a Perron-Frobenius matrix.*

**Proof.** Recall Theorem 2 of Lecture 2. For any pseudo-Anosov mapping class $\phi : S \to S$, there is a train track $\tau$ compatible with $\phi$ so that under iteration of $\phi$ any essential closed curve $\gamma$ on $S$ is eventually carried by $\tau$. That is, $\phi^k(\gamma)$ is carried by $\tau$ for large $k$. Furthermore, $\lambda$ is the growth rate of the length of $\phi^k(\gamma)$.

Let $V_\tau$ be the real vector space spanned by the edges of $\tau$. Let $W_\tau^* \subset V_\tau^*$ be the subspace of weight vectors $w \in V_\tau^*$ so that whenever $\vec{e}_1, \vec{e}_2, \vec{e}_3$ meet at a vertex, and $\vec{e}_1$ and $\vec{e}_2$ meet at a cusp, then

$$w(e_1) + w(e_2) = w(e_3).$$

Then $W_\tau*$ defines a linear subspace of $V_\tau^*$.

The integral points on the positive cone

$$C = \{w \in W_\tau* \ : \ w(e_i) \geq 0 \text{ for all } i\},$$

in $W_\tau*$ is in one-to-one correspondence with closed curves carried by $\tau$. By the Theorem 2 of Lecture 2, $(T^*)^k$ is eventually positive on $W_\tau*$. Thus $T_*|_{W_\tau*}$ is Perron-Frobenius.

The following is a standard fact from linear algebra. For any $n \times n$ matrix $F$,

$$\mathrm{Spec}(F) = \lim_{k \to \infty} |F^k v|^{\frac{1}{k}}, \tag{1}$$

where $|\ |$ is any norm on $\mathbb{R}^n$, and $v$ is a general vector (that is, $v$ is taken from the complement in $\mathbb{R}^n$ of a finite number of codimension $\geq 1$ hyperplanes in $\mathbb{R}^n$).

Since the vectors in $C$ span $W_\tau*$, we have

$$\mathrm{Spec}(T^*|_{W_\tau*}) = \lambda(\phi).$$

$\square$

**Proposition 2** *Let $F$ be a Perron-Frobenius $n \times n$ matrix. Then we have the following:*

*(i) there is a unique positive eigenvector $v \in \mathbb{R}^n$ up to positive scalar multiplication,*

*(ii) the eigenvalue $\lambda$ of $v$ is the spectral radius of $F$, and*

*(iii) for any other eigenvalue $\mu$ of $F$, $\lambda > |\mu|$.*

We include a proof of (i) and (ii) since it illustrates the close analogy between the behavior of pseudo-Anosov maps, and the behavior of Perron-Frobenius matrices.

**Proof of (i) and (ii).** The matrix $F$ defines a continuous function

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

preserving the origin, and hence it defines a function

$$\begin{aligned} f : S^{n-1} &\to S^{n-1} \\ z &\mapsto F(z)/|F(z)| \end{aligned}$$

where $S^{n-1}$ is identified with the elements of $\mathbb{R}^n$ with $|z| = 1$.

Let $R \subset S^n$ be the convex subspace corresponding to positive vectors in $\mathbb{R}^n$. Then since $F$ is non-negative, we have $f(R) \subset R$, and for $n$ large $f^n$ sends the closure of $R$ strictly inside $R$. It follows from the Brouwer fixed point theorem that $f$ has a unique fixed point $v_0$ in $R$, and

$$v_0 = \lim_{n \to \infty} f^n(v) \tag{2}$$

for any $v \in R$.

Since $v_0$ is a fixed point of $f$, it is necessarily an eigenvector of $F$, and is the unique positive eigenvector up to scalar multiplication. The eigenvalue $\lambda$ of $v_0$ is necessarily real and $> 1$, since both $v_0$ and $F(v_0)$ are positive vectors.

Finally, we show that $\lambda$ equals the spectral radius of $F$. By (2), for any $v \in R$, $|F^n(v)|^{\frac{1}{n}}$ tends to $\lambda$. Since the vectors in $R$ span $\mathbb{R}^n$, (ii) follows from (1). $\square$

**Remarks.**

1. The vector $v_0$ is called the *Perron-Frobenius eigenvector* of $F$, and will be useful again in a later lecture when we talk about Thurston's examples of pseudo-Anosov maps associated to bipartite graphs.
2. The vector $v_0$ also describes a natural generalization of simple closed curves carried on $\tau$ called a transverse measured foliation. This foliation is invariant under the action of $\phi$ and $\phi$ acts on it by stretching by a factor of $\lambda$.

**Perron units.** A monic integer polynomial $P(x)$ is called a *Perron polynomial* if it has a real root $\lambda > 1$ of multiplicity one, such that for any other root $\mu$ of $P$, $\lambda > |\mu|$. The largest root $\lambda$ of a Perron polynomial is called a *Perron number.*

**Proposition 3** *A monic polynomial $P(x)$ is a Perron polynomial if and only if there is a Perron-Frobenius matrix $F$ so that $P(x)$ is the characteristic polynomial of $F$.*

A polynomial $P(x)$ is called *reciprocal* if $P(x) = x^d P(\frac{1}{x})$ where $d$ is the degree of $P$. Roots of monic reciprocal polynomials are called *units.* One can show that the transition matrices associated to pseudo-Anosov mapping classes preserve a symplectic form, and that hence the characteristic polynomial $P(x)$ is reciprocal.

**Proposition 4** *The dilatation of a pseudo-Anosov map is a Perron unit.*

The converse is an open question.

**Question 5 (Thurston)** *Is it true that every Perron unit is the dilatation of a pseudo-Anosov mapping class?*

## References

[Mey] C. D. Meyer. *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, 2000.