The dynamics of mapping classes on surfaces

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5 Digraphs

In this lecture we study Perron-Frobenius matrices using digraphs, and deduce general properties of digraphs associated to pseudo-Anosov mapping classes. For the most part of this lecture, we will follow Birman's analysis in [Bir]. We conclude this lecture with some examples.

Digraphs. A *digraph* is a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} = \{v_1, \dots, v_n\}$ and directed edges $\mathcal{E}$. Given a digraph Γ, there is an associated non-negative matrix

$$D_\Gamma = [d_{i,j}],$$

where $d_{i,j} = k$ if there are k directed edges from v_i to v_j (we do not count negatively oriented edges). A *path* on a digraph is any word in the edges of $\mathcal{V}$ so that for any two consecutive edges ϵ_1 and ϵ_2, the endpoint of ϵ_1 is the initial point of ϵ_2.

Lemma 1 *The number of distinct paths from v_i to v_j of length k is the i, j entry of $(D_\Gamma)^k$.*

A digraph Γ is *strongly path-connected* if there is a path from any vertex to any other vertex of Γ. A *cycle* is a path whose endpoint equals its initial point. If Γ is a cycle, then its associated matrix D_Γ has finite order, and its characteristic polynomial is cyclotomic. By contrast, we have the following.

Proposition 2 *The matrix D_Γ is Perron-Frobenius if and only if Γ is strongly path-connected but not equal to a cycle.*

In this case, we will call Γ a *PF-digraph*, and its spectral radius is the *dilatation* of Γ.

A *linear subgraph* of a digraph is a disjoint union of cycles. The *characteristic polynomial* of a PF-digraph is the characteristic polynomial of the associated Perron-Frobenius matrix. The following is a useful theorem from graph theory used by Birman in the context of dilatations (see Theorem 2, [Bir], and references therein).

Theorem 3 (Coefficient Theorem for Digraphs) *Let Γ be a PF- digraph with n vertices, and let $\mathcal{L}_i = \cup_j L_{i,j}$ be the set of all linear subgraphs of Γ having precisely i vertices. For each $L_{i,j} \in \mathcal{L}_n$, let $m(L_{i,j})$ denote the number of cycles in $L_{i,j}$. Then the characteristic polynomial of Γ is*

$$\chi_\Gamma(x) = x^n + \sum_{i=1}^{n} b_i x^{n-i},$$

where

$$b_i = \sum_{L_{i,j} \in \mathcal{L}_i} (-1)^{m(L_{i,j})}.$$

Define the *out-degree* of a vertex $v \in \mathcal{V}$ to be the number of directed edges in $\mathcal{E}$ with initial point v. The *complexity* c of a digraph Γ is the sum of out-degrees minus the number of vertices. The *house* of a polynomial f is the maximum absolute value of roots of f.

Theorem 4 (Birman) *If Γ is a digraph associated to a pseudo-Anosov mapping class, and has complexity $c \leq 2$, then the dilatation λ is greater than or equal to the house of a polynomial of the form:*

$$\chi_{n,a} = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1,$$

where $2 \leq g \leq n \leq 3g - 3$ and $0 < a < n$.

The polynomials $\chi_{n,a}$ are called Lanneau-Thiffeault polynomials, because they arise in work of Lanneau and Thiffeault on minimum dilatation orientable pseudo-Anosov mapping classes [LT].

Examples of digraphs with small complexity.

1. Smallest spectral radius PF-digraph

In [Pen], Penner found a lower bound for dilatations of pseudo-Anosov mapping classes by noting that if the corresponding digraph Γ has n vertices, then for any vertex v_i of Γ, there are at least two paths of length n emanating from v_i on Γ, for otherwise Γ would be a cycle. It follows that the sum of entries in $(D_\Gamma)^n e_i$, must be at least two for each basis vector e_i. Therefore

$$\frac{|(D_\Gamma)^n v|}{|v|} \geq 2,$$

for any non-negative vector v, and thus the dilatation λ of Γ satisfies

$$\lambda \geq 2^{\frac{1}{n}}.$$

Proposition 5 (Penner's lower bound) *Let δ_n^Γ be the smallest dilatation for a PF-digraph with n vertices. Then we have*

$$\log(\delta_n^\Gamma) \geq \frac{\log(2)}{n}.$$

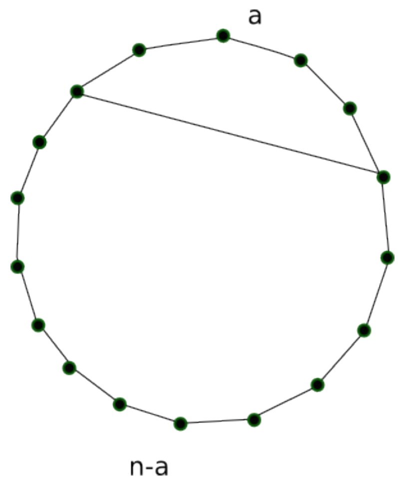

Figure 1: The smallest spectral radius PF-digraph.

The digraphs with $c = 1$ are shown in Figure 1. Here all edges are oriented in the same (say counter-clockwise) direction.

For these graphs there are exactly two cycles, one of length n and a cycle of length $n - a$, and there are no disjoint pairs of cycles. Thus, Theorem 3 implies that the characteristic polynomial is given by

$$\chi_n(x) = x^n - x^a - 1.$$

For fixed a, the largest root λ_n of χ_n has the property that $\lambda_n \to 1$ as $n \to \infty$. Moreover, $(\lambda_n)^n$ is bounded by $(1 + \epsilon)^a + 1$, where ϵ can be made arbitrarily small. This yields the following stronger version of Proposition 5 (cf. [McM] p. 44 and Figure 6).

Proposition 6

$$\lim_{n\to\infty} (\delta_n^\Gamma)^n = 2.$$

In fact, one can also verify that for fixed n, the smallest dilatation δ_n^Γ is obtained by setting $a = 1$.

2. PF-digraphs of complexity 2

In [Bir], Birman proved that for complexity $c = 2$, the types of PF-digraphs shown in Figure 2 have smallest spectral radius.

Using Theorem 3, one can check that the characterisitc polynomial for the digraph in Figure 2 is equal to

$$\chi_{m,b}(x) = x^{2m} - x^{2m-b} - x^m - x^b + 1.$$

This polynomial is known as an LT-poliynomial after Lanneau and Thiffeault [LT].

The polynomials have the following asymptotic behavior.

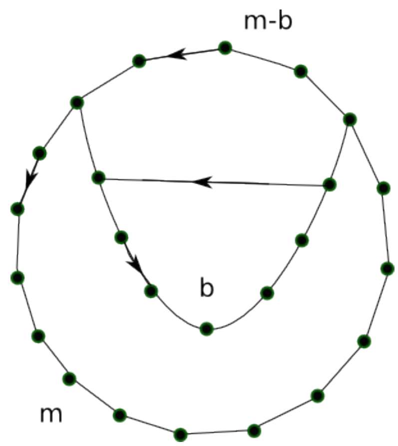

Figure 2: Minimizing digraphs for complexity $c = 2$.

Proposition 7 *For fixed b we have*

$$\log(\lambda_{m,b}) \asymp \frac{\log(m)}{m}.$$

Let $a = m - b$. Then

$$\chi_{m,m-a}(x) = x^{2m} - x^{m+a} - x^m - x^{m-a} + 1.$$

Proposition 8 *For fixed a we have*

$$\log(\lambda_{m,m-a}) \asymp \frac{1}{m},$$

and more precisely

$$\lim_{m\to\infty} (\lambda_{m,m-a})^m = \frac{3+\sqrt{5}}{2}.$$

Remark. There is no known upper bound for the complexity of digraphs associated to minimum dilatation pseudo-Anosov mapping classes. However, the following is a useful Lemma in this direction.

Lemma 9 (Ham-Song[HS]) *Any PF-digraph Γ has complexity c satisfying the inequality*

$$c \leq \lambda^n - 1,$$

where λ is the dilatation of Γ, and n is the number of vertices of Γ.

For a fixed surface, there is an upper bound on the minimal number of vertices of a digraph corresponding to a pseudo-Anosov mapping class. For a closed genus g surface, for example, the bound is $6g-6$. Even if we restrict to digraphs whose normalized dilatation is less than or equal to $\frac{3+\sqrt{5}}{2}$, this means that for closed surfaces, the complexity c satisfies

$$c \leq 6.$$

The number of possible digraphs is too large for current programs to handle even for $g=3$ ($n \leq 12$), which is the first genus where the minimum dilatation is not known.

References

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