The dynamics of mapping classes on surfaces

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6 Fibered Faces of 3-manifolds

In this lecture we briefly describe the theory of fibered faces of 3-manifolds. Our main sources are [Thu], [Fri] and [McM]. I will assume some knowledge of basic algebraic topology, including fundamental groups, covering spaces, and singular homology and cohomology.

A 3-manifold manifold M is *hyperbolic* if it admits a metric of constant sectional curvature -1, or equivalently it is quotient of the real hyperbolic 3-space by a discrete group of isometries. Unless otherwise stated, we will always assume that M is hyperbolic.

Fibrations of 3-manifolds. Let M be a compact oriented 3-manifold possibly with boundary components.

A 3-manifold M is *fibered* if there is a fibration $\psi : M \to S^1$ of M over the circle S^1. The map ψ is called a *fibration.* Two fibrations ψ_1, ψ_2 are equivalent if there is an isotopy

$$\mathfrak{h} : M \times [0,1] \to S^1,$$

so that

$$\begin{array}{rcl} \mathfrak{h}_t : M & \to & S^1 \\ x & \mapsto & h(x,t) \end{array}$$

is a fibration for each t. Given M, we denote by $\Psi(M)$ the set of all fibrations of M up to this equivalence.

Recall that to each manifold X (or more generally, any CW-complex) we can associate an abelian group $H_1(X,\mathbb{Z})$, called the *first homology group* of X. We can think of $H_1(X;\mathbb{Z})$ as the group of free homotopy classes of oriented closed curves on X. Two curves are homologically equivalent if their difference bounds a subsurface of X.

If X Is connected, the first homology group $H_1(X;\mathbb{Z})$ is closely related to the fundamental group $\pi_1(X,x_0)$, where $x_0 \in X$ is any point. The Hurewicz map

$$\pi_1(X,x_0) \to H_1(X;\mathbb{Z})$$

is the map sending each basepointed loop in $\pi_1(X, x_0)$ to the homotopy type of the loop ignoring the basepoint. By Hurewicz' theorem, if X is connected, then this map is surjective, and $H_1(X; \mathbb{Z})$ is the abelianization of $\pi_1(X, x_0)$.

An arc on X is *relatively closed* if its endpoints lie on the boundary of X. Two relatively closed curves are homologically equivalent if they bound a subsurface of X that may include subarcs of the boundary of X. Thus, for example, a closed curve that is the boundary of an annular neighborhood of a boundary component is trivial in the relative homology.

Example. If S is a surface of genus g and n boundary components, then $H_1(S; \mathbb{Z})$ is freely generated by $2g + n - 1$ simple closed curves. On the other hand, $H_1(S, \partial S; \mathbb{Z})$ is freely generated by $2g$ simple closed curves, and $n - 1$ relatively closed curves $\gamma_1, \ldots, \gamma_{n-1}$, where under some ordering of the boundary components $b_1, \ldots, b_n$ of S, γ_i is an arc from b_i to b_{i+1}.

Homology groups are functorial. Thus, given a fibration $\psi : M \to S^1$, there is a corresponding map on homology:

$$\psi_* : H_1(M; \mathbb{Z}) \quad \to \quad H_1(S^1; \mathbb{Z}). \tag{1}$$

For the purposes of this lecture, it is enough to think of the integral cohomology group $H^1(M; \mathbb{Z})$ as the group of homomorphisms

$$H_1(M; \mathbb{Z}) \to \mathbb{Z}.$$

Then, since $H_1(S^1; \mathbb{Z}) = \mathbb{Z}$, by (1) each element of $\Psi(M)$ determines an element of $H^1(M; \mathbb{Z})$, and this element is unique. Thus we can consider $\Psi(M)$ as a subset of $H^1(M; \mathbb{Z})$.

Let $H^1(M; \mathbb{R}) = H^1(M; \mathbb{Z}) \otimes \mathbb{R}$. This vector space can be interpreted using, for example, the language of differential one-forms, but for our purposes we will consider $H^1(M; \mathbb{R})$ as the collection of homomorphisms from $H_1(M; \mathbb{Z})$ to $\mathbb{R}$. Then we have

$$\Psi(M) \subset H^1(M; \mathbb{Z}) \subset H^1(M; \mathbb{R}) = \mathbb{R}^{b_1},$$

where b_1 is the first Betti number of M.

Thurston's theory of fibered faces gives a concrete way to view the locus $\Psi(M)$ as a subset of $H^1(M; \mathbb{R})$. Given a connected surface S, let

$$\chi_-(S) = \max\{-\chi(S), 0\}.$$

Given a surface $S = \cup_{i=1}^k S_i$, where S_i are connected disjoint subsurfaces, let $\chi_-(S) = \sum_{i=1}^k \chi_-(S_i)$. Let $\psi \in H^1(M; \mathbb{Z})$. Let

$$||\psi|| = \min\{\chi_-(S) \ : \ \psi|_{i_*(H_1(S;\mathbb{Z}))} = 0\}.$$

where S ranges amongst subsurfaces of M, $i : S \to M$ is inclusion, and $i_*(H_1(S; \mathbb{Z}))$ is the image of the induced map $i_* : H_1(S; \mathbb{Z}) \to H_1(M; \mathbb{Z})$.

Theorem 1 (Thurston [Thu]) *The function $||\ ||$ extends to a norm on $H_1(M; \mathbb{R})$, whose unit norm ball B is a convex polyhedron. For each top-dimensional face F of B, let $C_F = F \cdot \mathbb{R}^+$ be the positive cone over F. Then one of the following is true:*

1. $C_F \cap \Psi(M) = \emptyset$*, or*

2. $C_F \cap \Psi(M) = C_F \cap H^1(M;\mathbb{Z})$.

If $C_F \cap \Psi(M) = C_F \cap H^1(M;\mathbb{Z})$, then we say that F is a *fibered face* of M. Theorem 1 implies that if one integral element of C_F corresponds to a fibration of M, then so do all the integral elements of C_F. Let $\Psi(M,F) = C_F \cap \Psi(M)$.

Mapping classes parameterized by fibered faces. Given a mapping class $\phi : S \to S$, the *mapping torus* M_ϕ of ϕ is the three manifold obtained by taking the quotient

$$M_\phi = S \times [0,1]/(x,1) \sim (\phi(x),0).$$

A 3-manifold M is fibered over S^1 if and only if $M = M_\phi$ for some ϕ. In this case, we say ϕ is the *monodromy* of the fibration. Let $\Phi(M)$ be the collection of monodromies of fibrations of a 3-manifold M. Then $\Psi(M)$ and $\Phi(M)$ are in canonical one-to-one correspondence, and each partitions into a union of $\Psi(M,F)$, respectively, $\Phi(M,F)$.

Deformations of pseudo-Anosov mapping classes

Pseudo-Anosov mapping classes and hyperbolic 3-manifolds are related in the following way.

Theorem 2 (Thurston) *A mapping class $\phi : S \to S$ is pseudo-Anosov if and only if M_ϕ is hyperbolic.*

Thus, the set of pseudo-Anosov mapping classes can be partitioned into sets of the form $\Phi(M,F)$, where M is a hyperbolic 3-manifold, and F is a fibered face of M, and the $\Phi(M,F)$ can be thought of as equivalence classes. Let $\phi : S \to S$ be a fixed mapping class and assume $\phi \in \Phi(M,F)$. Then we can think of the other elements of $\Phi(M,F)$ as deformations of ϕ. Identify $\Phi(M,F)$ with $\Psi(M,F)$, and define the normalized dilatation of $\phi \in \Phi(M,F)$ to be

$$\overline{\lambda}(\phi) = \lambda(\phi)^{||\psi||}.$$

An element of $H^1(M;\mathbb{Z})$ is *primitive* if it is not the multiple of another element of $H^1(M;\mathbb{Z})$ other than itself. An element $\psi \in \Psi(M)$ is primitive as an element of $H^1(M;\mathbb{Z})$ if and only if the fibers of ψ are connected. More generally, we have the following.

Lemma 3 *The number of connected components of the fiber of ψ equals the index of the image of ψ_* in $H_1(S^1;\mathbb{Z})$. In particular, the fibers of ψ are connected if and only if ψ_* is surjective.*

Let $\psi \in \Psi(M,F)$, r a positive integer, and $\psi_r = r\psi$. If ψ is primitive, then $(\psi_r)_*$ is the map $(\psi)_*$ composed with

$$\begin{array}{ccc} \mathbb{Z} & \to & \mathbb{Z} \\ a & \mapsto & ra \end{array}$$

Thus, the fiber of $r\psi$ considered as a fibration has r connected components.

Geometrically, the fiber of ψ_r is the union of r copies of the fiber of ψ, and the monodromy ϕ_r of ψ_r restricted to each copy has the property that $(\phi_r)^r = \phi$. Thus,

$$\lambda(\phi_r) = \lambda(\phi)^{\frac{1}{r}}.$$

Since $||r\psi|| = |r|||\psi|| = r||\psi||$, it follows that

$$\overline{\lambda}(\phi_r) = \lambda(\phi_r)^{||\psi_r||} = \lambda(\phi)^{||\psi||} = \overline{\lambda}(\phi).$$

For each $\psi \in \Psi(M, F)$, let $\overline{\psi} = \frac{1}{||\psi||}\psi$. This is the intersection of the ray containing ψ with F. Then we immediately have the following:

- the primitive elements of $\Psi(M, F)$ are in 1-1 correspondence with rational points on F,
- the least common multiple of the denominator of $\overline{\psi}$ is the Thurston norm of the corresponding primitive element in the ray defined by ψ, and
- $\overline{\lambda}$ is a well-defined function on the rational points of F.

Theorem 4 (Fried [Fri], McMullen [McM]) *The map*

$$\begin{aligned} K : \Psi(M, F) &\to \mathbb{R} \\ \psi &\mapsto \overline{\lambda}(\phi) \end{aligned}$$

extends to a convex function on F that is constant on rays through the origin, and goes to infinity as ψ approaches the boundary of F.

Corollary 5 *The normalized dilatation has a unique minimum on F.*

Let G be a group and $f \in \mathbb{Z}G$ an element of the group ring. Then

$$f = \sum_{g \in G} a_g g,$$

where $a_g = 0$ except for a finite subcollection of $g \in G$. Take any homomorphism $\psi : G \to \mathbb{Z}$. Then the *specialization* of f to ψ is the polynomial

$$f^{\psi}(x) = \sum_{g \in G} a_g x^{\psi(g)}.$$

Theorem 6 (McMullen [McM]) *Let M be a hyperbolic 3-manifold, F a fibered face, and let $G = H_1(M; \mathbb{Z})$. Then there is an element $\Theta \in \mathbb{Z}G$ such that for all $\psi \in \Psi(M, F)$, $\lambda(\psi)$ is the largest root of $\Theta^{\psi}(x)$.*

The element Θ is called the *Teichmüller polynomial* of (M, F).

Example: The simplest pseudo-Anosov map, and the $\mathbf{6_2^2}$ link complement.

The simplest pseudo-Anosov map ϕ_0 has mapping torus equal to the complement in S^3 of the link shown in Figure 1.

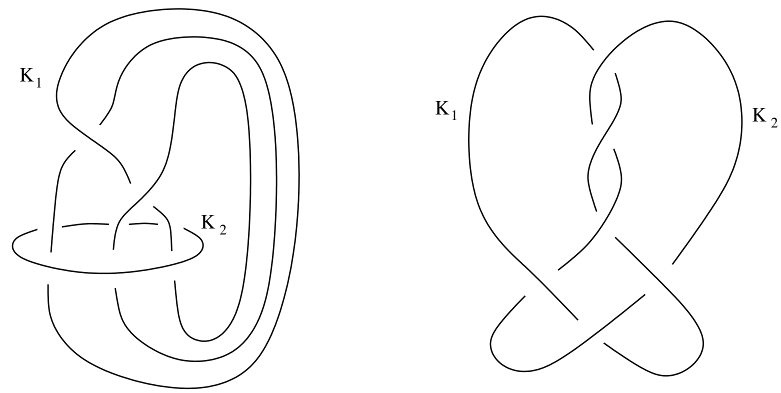

Figure 1: The 6_2^2 link.

The first Betti number, or dimension of $H_1(M;\mathbb{Z})$ equals 2. Thus, the fibered face containing ϕ_0 is one-dimensional. Figure 2 gives an illustration of the fibered cone.

Proposition 7 *There are generators t, u of $H_1(M;\mathbb{Z})$ so that the Teichmüller polynomial for the fibered face (M, F) containing the simplest pseudo-Anosov map is*

$$\Theta = u^2 - u(1 + t + t^{-1}) + 1.$$

Corollary 8 *The dilatation of the mapping class $\phi_{a,n}$ is the largest root of the equation*

$$P_{(a,n)}(x) = x^{2n} - x^{n+a} - x^n - x^{n-a} + 1.$$

References

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[Thu] W. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.* **339** (1986), 99–130.

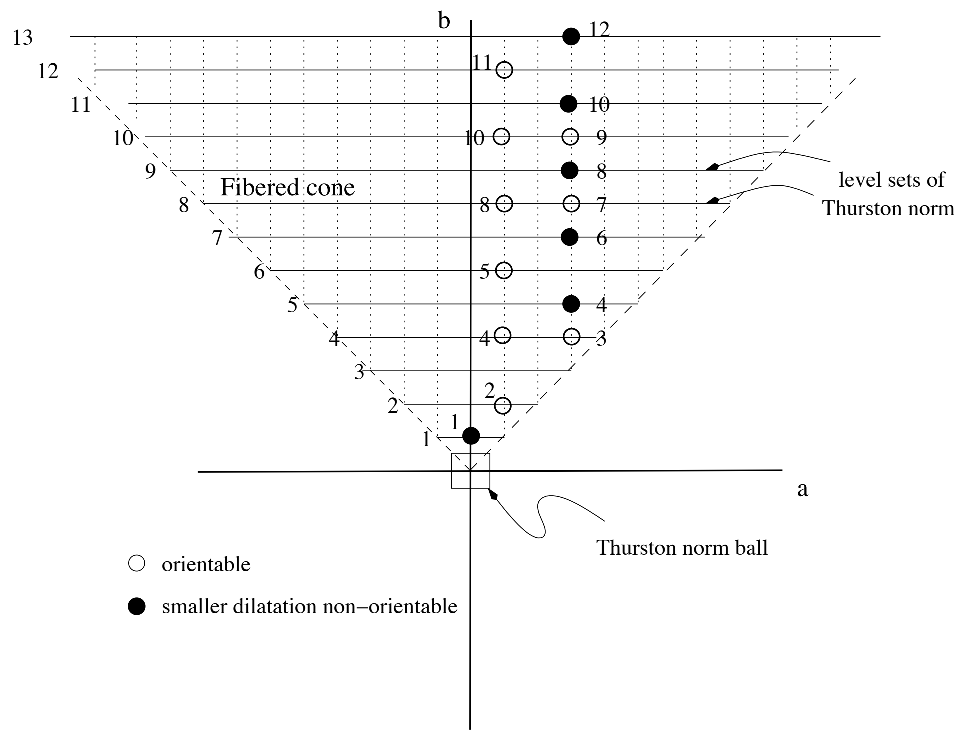

Figure 2: Fibered cone containing the simplest pseudo-Anosov braid.