The dynamics of mapping classes on surfaces

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January 20, 2012

7 Fibered faces and deformations of mapping classes

Let M be a hyperbolic 3-manifold. Let $\phi : S \to S$ be a pseudo-Anosov mapping class such that $M = M_\phi$ is the mapping torus. We have seen that there is a fibered face $F \subset H^1(M;\mathbb{R})$ so that ϕ is the monodromy of ψ for some integral point ψ in C_F. Furthermore, Fried and McMullen's theorems show that the normalized dilatation of mapping classes that arise as the monodromy of integral points on C_F defines a convex function on F. In the remaining lectures, we will investigate the problem: describe the monodromies $\Phi(M,F)$ corresponding to rational points on F in terms of the monodromy ϕ. We start in this lecture by describing the monodromies using the theory of covering spaces. This point of view can also be found in [McM].

The idea is that $\Psi(M,F)$ is a subset of $H^1(M;\mathbb{Z})$ which in turn can be identified with

$$\mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}).$$

Let $\widetilde{M}^{\mathrm{ab}}$ be the maximal abelian covering of M. Then the group of covering automorphisms is isomorphic to the first homology group of M:

$$\mathrm{Aut}_M(\widetilde{M}^{\mathrm{ab}}) \simeq H_1(M;\mathbb{Z}).$$

The hyperbolic structure on M lifts to $\widetilde{M}^{\mathrm{ab}}$ and is preserved by the action of $H_1(M;\mathbb{Z})$.

The fibration $\psi : M \to S^1$ and monodromy $\phi : S \to S$, defines a pseudo-Anosov flow on M. This lifts to define a flow on $\widetilde{M}^{\mathrm{ab}}$ Furthermore, S lifts to a surface $\widetilde{S}$ (of infinite type) in $\widetilde{M}$ that is transverse to the flow and the monodromy ϕ can be seen as the effect of translating S by a unit length along the flow. If ψ' is another fibration in $\Psi(M,F)$, then to understand the monodromy of ψ', it is enough to find a surface that is transversal to the original flow, that is also preserved by the action of the kernel of ψ' considered as a homomorphism $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})$.

Abelian coverings. What follows can be found in any introductory textbook in algebraic topology. Assume throughout that all spaces are path connected and semi-locally simply-connected. An (unbranched) covering $X \to Y$ is *abelian* if it is a regular covering with abelian group of covering autmorphisms $\mathrm{Aut}_Y(X)$. The set of abelian coverings is in 1-1 correspondence is in 1-1 correspondence with epimorphisms

$$H_1(X;\mathbb{Z}) \to G,$$

and G can be identified with $\mathrm{Aut}_Y(X)$.

If $\rho : X \to Y$ is a covering and $Z \to Y$ is a continuous map from a path connected space, then there is a lift $f' : Z \to X$ so that the diagram

$$\begin{tikzcd} & X \arrow[d, "\rho"] \\ Z \arrow[ur, "f'"] \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes if and only if

$$f_*\pi_1(Z) \subset \rho_*\pi_1(X).$$

Exercise. If the covering $\rho : X \to Y$ is abelian, show that

$$f_* H_1(Z) \subset \rho_*(H_1(X))$$

is sufficient for f to lift to X.

Given $\phi : S \to S$, and $\psi : M \to S^1$ the corresponding fibration of $M = M_\phi$, the map ψ induces an epimorphism

$$\psi_* : \pi_1(M) \to \mathbb{Z},$$

and hence a cyclic covering

$$\begin{tikzcd} M_\psi \arrow[r, "="] \arrow[d] & S \times \mathbb{R} \arrow[dl] \\ M & \end{tikzcd}$$

The group of covering transformations is generated by

$$\begin{aligned} T_\phi : S \times \mathbb{R} &\to S \times \mathbb{R} \\ (x, t) &\mapsto (\phi(x), t - 1). \end{aligned}$$

Then T_ϕ generates the covering automorphisms $\mathrm{Aut}_M(M_\psi)$.

Furthermore, the correspondence

$$\Psi(M, F) \quad \longrightarrow \quad \{\text{cyclic coverings } M_\psi \to M\}$$

is one-to-one.

Now consider the covering defined by

$$H_1(M; \mathbb{Z}) \xrightarrow{\text{id}} H_1(M; \mathbb{Z}).$$

This defines the *maximal abelian covering*

$$\begin{tikzcd} \widetilde{M}^{\text{ab}} \arrow[loop right, "G"] \arrow[d] \\ M. \end{tikzcd}$$

whose group of covering automorphism is $G = H_1(M; \mathbb{Z})$.

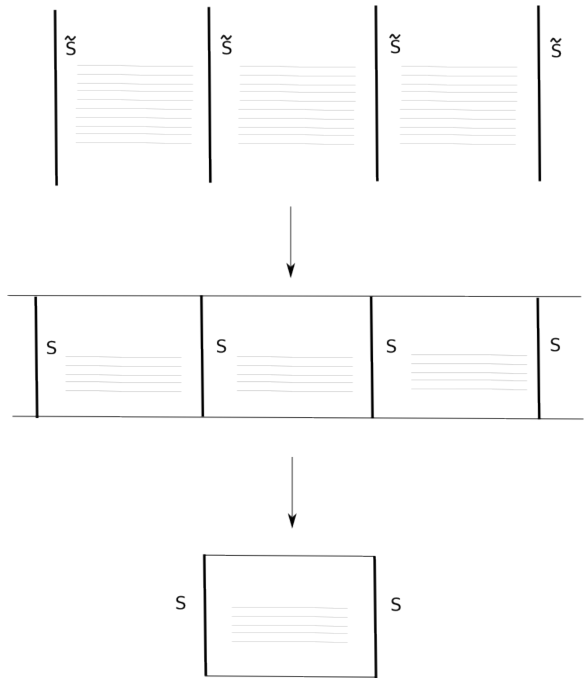

Figure 1: Tower of abelian coverings

Lemma 1 *For any abelian covering $M' \to M$ there is a factorization*

$$\begin{tikzcd} \widetilde{M}^{ab} \arrow[dd] \arrow[dr] & \\ & M' \arrow[dl] \\ M & \end{tikzcd}$$

where all arrows are covering maps.

Lemma 2 *There is a one-to-one correspondence between intermediate coverings of $\widetilde{M}^{ab} \to M$, and subgroups of $G = H_1(M;\mathbb{Z})$.*

Proof. Recall that $H_1(M;\mathbb{Z}) == \mathrm{Aut}_M(\widetilde{M}^{\mathrm{ab}})$. Thus, a subgroup K of G determines and intermediate cover

$$\begin{tikzcd} M' = \widetilde{M}^{\mathrm{ab}}/K \arrow[d] \\ M, \end{tikzcd}$$

and if $M' \to M$ is defined by $\psi' : \pi_1(M) \to G'$, then setting $K = \ker(\psi')$ we see that the correspondence is 1-1 and onto. $\square$

Now take $\psi \in \Psi(M, F)$ and consider $M_\psi \to M$. Then $M_\psi = \widetilde{M}^{\mathrm{ab}}/K_\psi$ where $K_\psi = \ker(\psi)$. Here ψ can be considered as

$$\begin{aligned} \psi : H_1(M) &\to \langle T_\phi \rangle \\ \gamma &\mapsto T_\phi^{m_S(\gamma)} \end{aligned}$$

where $m_S(\gamma)$ is the algebraic intersection of γ with S.

For $\alpha \in H^1(S;\mathbb{Z})$, we say α is *phi*-invariant if $\alpha \circ \phi_* = \alpha$. Let $\beta_1, \ldots, \beta_k$ be generators for $H^1(S;\mathbb{Z})^{\phi-\mathrm{inv}}$. Let

$$\begin{aligned} \underline{\beta} : H_1(S;\mathbb{Z}) &\to \mathbb{Z}^k \\ \gamma &\mapsto (\beta_1(\gamma), \ldots, \beta_k(\gamma)). \end{aligned}$$

Let $\rho : \widetilde{S} \to S$ be the corresponding covering, and let $h_1, \ldots, h_k$ generate $\mathrm{Aut}_S(\widetilde{S}) = \mathbb{Z}^k$.

Lemma 3 *The map ϕ lifts to $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$, i.e., there is a commutative diagram*

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\rho"'] & \widetilde{S} \arrow[d, "\rho"] \\ S \arrow[r, "\phi"] & S, \end{tikzcd}$$

and $\widetilde{\phi}$ is defined up to elements of $Aut_S(\widetilde{S})$.

Proof. By the Exercise, since ρ is an abelian covering, it is enough to check that $(\phi \circ \rho)_*(\pi_1(\widetilde{S})) \subset \rho_*(\widetilde{S})$. Since $\underline{\beta} \circ \phi_* = \underline{\beta}$, we have $\gamma \in \ker(\underline{\beta})$ if and only if $\phi_*(\gamma) \in \ker(\underline{\beta})$. Thus,

$$\begin{aligned} \rho_*(H_1(\widetilde{S})) &= \ker(\underline{\beta}) \\ &= \phi_*(\ker(\underline{\beta})) \\ &= \phi_* \rho_*(H_1(\widetilde{S})). \end{aligned}$$

□

The maximal abelian covering $\widetilde{M}^{\mathbf{ab}}$. Let $H_1, \ldots, H_k$ be defined by

$$\begin{aligned} H_i : \widetilde{S} &\to \widetilde{S} \\ (x, t) &\mapsto (h_i(x), t) \end{aligned}$$

for $i = 1, \ldots, k$, and

$$\begin{aligned} T_{\widetilde{\phi}} : \widetilde{S} &\to \widetilde{S} \\ (x, t) &\mapsto (h_i(x), t) \end{aligned}$$

□

Exercise. Show that $H_1(M;\mathbb{Z})$ is freely generated by $T_{\widetilde{\phi}}$ and $H_1,\ldots,H_k$.

Deformations of fibrations and mapping classes. Let $\psi' \in \Psi(M,F)$ and consider ψ' as a homomorphism

$$\psi' : H_1(M;\mathbb{Z}) \to \mathbb{Z}.$$

Let $K_{\psi'} = \ker(\psi')$. Then we have a commutative diagram:

$$\begin{tikzcd}
\widetilde{M}^{\mathrm{ab}} \arrow[loop left, "T_{\widetilde{\phi}'}"] \arrow[r, "="] \arrow[d] & \widetilde{S} \times \mathbb{R} \arrow[d] \\
M_{\psi'} \arrow[loop left, "T_{\phi'}"] \arrow[r, "="] \arrow[d] & S \times \mathbb{R} \arrow[dl] \\
M &
\end{tikzcd}$$

Lemma 4 *The covering map $T_{\widetilde{\phi}'} : \widetilde{M}^{ab} \to M^{ab}$ defined by the lift $\widetilde{\phi}'$ of ϕ' is any solution*

$$T \in H_1(M;\mathbb{Z}) = Aut_M(\widetilde{M}^{ab}$$

such that

$$\psi'(T) = -1.$$

By this lemma, one technique for describing $\phi' : S' \to S'$ is to find a surface $\widetilde{S}' \subset \widetilde{M}^{\mathrm{ab}}$ that is transversal to the pseudo-Anosov flow defined by ψ and ϕ that is left invariant under the action of $K_{\psi'}$.

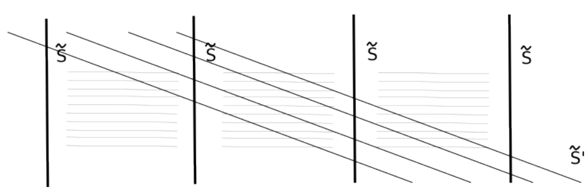

Figure 2: A deformation of the fiber.

References

[McM] C. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. *Ann. Sci. École Norm. Sup.* **33** (2000), 519–560.