The dynamics of mapping classes on surfaces

Eriko Hironaka

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8 Teichmüller polynomials, fibered faces, and families of digraphs.

Let M be a hyperbolic 3-manifold, F a fibered face, $G = H_1(M;\mathbb{Z})$. Recall that the dimension of F is $b_1(M) - 1$, and rational points on F are in one-to-one correspondence with primitive elements of $\Psi(M, F)$, the fibrations with connected fibers, and primitive elements of $\Phi(M, F)$.

Given an element $f = \sum_{g \in G} a_g g \in \mathbb{Z}G$, and $\psi \in H^1(M;\mathbb{Z})$, the *specialization* of f by ψ is given by

$$f^\psi(x) = \sum_{g \in G} a_g x^{\psi(g)}.$$

Given a polynomial $f(x)$, the *house* $|f|$ of f is the maximum norm amongst roots of f.

Theorem 1 (McMullen [McM]) *There is an element $\Theta \in \mathbb{Z}G$ such that for all $\psi \in \Psi(M, F)$ with monodromy $\phi : S \to S$,*

$$\lambda(\phi) = |\Theta^\psi|.$$

In this lecture, we will show how traintracks for mapping classes in $\Phi(M, F)$ are related to one another.

As before, let $\psi \in \Psi(M, F)$, and let $\phi : S \to S$ be the monodromy. Let $\beta_1, \ldots, \beta_k$ be generators for $H^1(S;\mathbb{Z})^{\psi-\mathrm{inv}}$. Let $\underline{\beta} : H_1(S;\mathbb{Z}) \to \mathbb{Z}^k$, be the corresponding epimorphism, and let

$$\widetilde{\rho} : \widetilde{S} \to S,$$

be the corresponding $\mathbb{Z}^k$-coveirng of S.

Identify $\widetilde{M}^{\mathrm{ab}}$ with $\widetilde{S} \times \mathbb{R}$. Let $\zeta_1, \ldots, \zeta_k$ be generators of $\mathrm{Aut}_S(\widetilde{S})$, and let

$$Z_i(x, s) = (\zeta_i(x), s),$$

for all $(x, t) \in \widetilde{S} \times \mathbb{R}$. Choose a lift $\widetilde{\phi}$ of ϕ, and let

$$T(x, s) = (\widetilde{\phi}, s - 1).$$

Let $\tau \subset$ be a train track for ϕ and let $\widetilde{\tau}$ be its preimage in $\widetilde{S}$.

Lemma 2 *the lifted train track $\widetilde{\tau}$ carries $\widetilde{\phi}$, and is invariant under the action of $G = H_1(M;\mathbb{Z})$ on $\widetilde{M}^{ab}$.*

Let V_τ be the vector space spanned by the edges of τ. The integral points of V_τ (elements with integral coefficients), correspond to $\mathbb{Z}$-linear combinations of the edges of τ. Let $V_{\widetilde{\tau}}$ be the free $\Lambda_k = \mathbb{R}\mathbb{Z}^k$ module spanned by the edges of τ. The integral points of $V_{\widetilde{\tau}}$ can be thought of as $\mathbb{Z}$-linear combinations of the edges of $\widetilde{\tau}$, respecting the action of $\mathbb{Z}^k$.

Let $\Theta \in \mathbb{Z}G$ be the characteristic polynomial of the transition matrix for the train track map

$$A_{\widetilde{\phi}} : V_{\widetilde{\tau}} \to V_{\widetilde{\tau}}$$

determined by $\widetilde{\phi}$.

The ring Λ_k can be identified with the ring of Laurent polynomials

$$\Lambda_k = \mathbb{R}[t_1^{\pm 1}, \dots, t_k^{\pm 1}, u],$$

where $t_1, \dots, t_k$ are associated to the covering transformations $Z_1, \dots, Z_k \in \mathrm{Aut}_S(\widetilde{S})$ and u is associated to the map $T_{\widetilde{\phi}}$. Thus Θ can be thought of as a Laurent polynomial, and we will call it the *Teichmüller polynomial* for the fibered face F. (This is slightly different from McMullen's definition.) The elements of $G = H_1(M;\mathbb{Z})$ correspond to monomials in $t_1, \dots, t_k$ and u.

Consider the suspension $E_{\widetilde{\tau}}$ of $\widetilde{\tau}$ in $\widetilde{S} \times \mathbb{R}$. This covers the suspension E_τ of τ in M. The map $T_{\widetilde{\phi}}$ defines a map of $E_{\widetilde{\tau}}$ sending rectangles $\epsilon \times [0,1]$, where ϵ is an edge to $\sigma \times [0,1]$, where σ is an edge-path on τ. We will call such maps *folding maps*. Letting $Z_1, \dots, Z_k$ act as homeomorphisms (with no folding) on $E_{\widetilde{\tau}}$, we have a group homomorphism

$$G \to \mathcal{F}(E_{\widetilde{\tau}})$$

from G to folding maps on $E_{\widetilde{\tau}}$.

Let $\psi_1 \in \Psi(M, F)$, and $\phi_1 : S_1 \to S_1$ be its monodromy. We can construct the covering $\widetilde{S}_1 \to S_1$. Then the lift of $\widetilde{\phi}_1$ to $\widetilde{S}_1$ determines a new transformation

$$T_{\widetilde{\phi}_1}(x_1, s_1) = (\widetilde{\phi}_1(x_1), s_1),$$

with respect to the new coordinates (x_1, s_1) defined by identifying $\widetilde{M}^{\mathrm{ab}}$ with $\widetilde{S}_1 \times \mathbb{R}$. The transformation $T_{\widetilde{\phi}_1}$ is an element of $\mathrm{Aut}_M(\widetilde{M}^{\mathrm{ab}} = G$, determined up to multiplication by the subgroup of G generated by the image of $\mathrm{Aut}_{S_1}(\widetilde{S}_1)$ in G.

Identify $\widetilde{S}_1$ with $\widetilde{S}_1 \times \{0\} \in \widetilde{S}_1 \times \mathbb{R} = \widetilde{M}^{\mathrm{ab}}$. Then $\widetilde{\phi}_1 : \widetilde{S}_1 \times \widetilde{S}_1$ is the mapping class defined by applying an element $T \in G$ such that $\psi_1(T) = -1$ to $\widetilde{S}_1 \times \{0\}$ and then flowing back to $\widetilde{S}_1 \times \{0\}$. One sees that in order for the flow back to be well-defined we need ψ_1 to be "close" to ψ.

Magic braid, and its monodromy

The *magic manifold* is the mapping torus for the braid monodromy defined by $\sigma_1\sigma_2^{-1}\sigma_1$ in terms of the usual braid generators. The braid is drawn in Figure 1.

Figure 1: Braid defining the magic manifold.

In Figure 2, we see a train track map for the monodromy, and its lift to $\widetilde{S}$. The corresponding train track map on $V_{\widetilde{\tau}}$ is given by the matrix

$$\begin{bmatrix} s(1+t) & s(1+t+t^{-1}) \\ 1 & 1+t^{-1} \end{bmatrix}.$$

Thus, the Teichmüller polynomial is given by

$$\Theta(u,t) = u^2 - u(1+t^{-1}+s+st)+s.$$

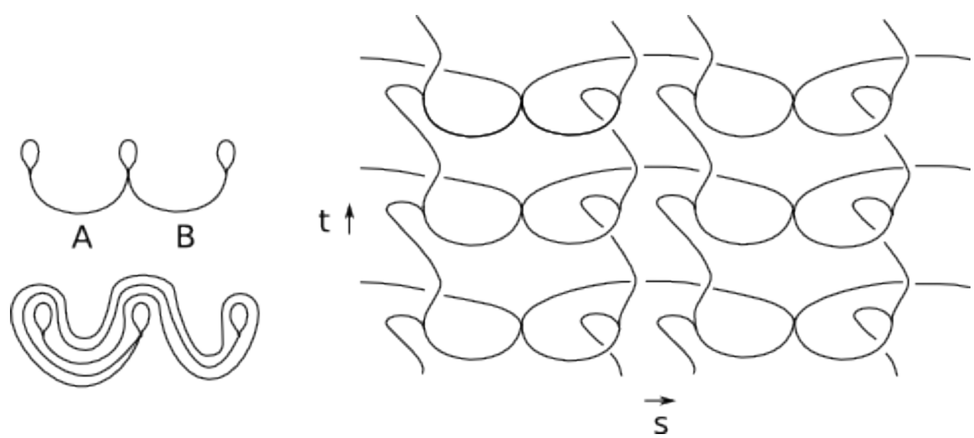

Figure 2: Train track map for the monodromy, and its lift to covering.

Exercise. Choose lifts $\widetilde{A}$ and $\widetilde{B}$ on $\widetilde{\tau}$ lifting A and B on τ, and draw their images under $\widetilde{\phi}$. (Hint. Find a choice of $\widetilde{A}$, $\widetilde{B}$, and lifting $\widetilde{\phi}$ of ϕ so that the images of $\widetilde{A}$ and $\widetilde{B}$ can be drawn on Figure 2).

Train tracks and digraphs in the simplest pseudo-Anosov map example. Let $\phi : S \to S$ be the simplest pseudo-Anosov braid, $M = M_\phi$, and $\psi : M \to S^1$ the corresponding fibration. Let $\alpha \subset S$ be a representative for the generator of $H_1(S, \partial S; \mathbb{Z})$, and let $\widetilde{S} \to S$ be the corresponding cyclic covering. Let $\xi \in H^1(M;\mathbb{Z})$ be the dual to the suspension of α in M. Then ψ and ξ generate $H^1(M;|Z)$.

We will now consider the elements $\psi_n = n\psi + \xi \in H^1(M;\mathbb{Z})$. These define rays in $H^1(M;\mathbb{Z})$ through the origin that converge to the ray through ψ. Thus, for large n, ψ_n is in the fibered face

F such that $\psi \in \Psi(M, F)$, and if ϕ_n is the monodromy of ψ_n, then

$$\overline{\lambda}(\phi_n) \to \overline{\lambda}(\phi) = \left(\frac{3+\sqrt{5}}{2}\right)^2.$$

The kernel of ψ_n is generated by $Z^n T$, where Z corresponds to a generator of $\mathrm{Aut}_S(\widetilde{S})$, and $T = T_{\widetilde{\phi}}$. Thus, $\widetilde{S}_n$ is a surface in $\widetilde{M}^{\mathrm{ab}}$ that is invariant under the action of $Z^n T$. Furthermore, $\psi_n(Z) = -1$ for all n. Thus, $Z = T_{\widetilde{\phi}_n}$.

Here is a picture of the train tracks and train track maps for $\widetilde{S}_n$.

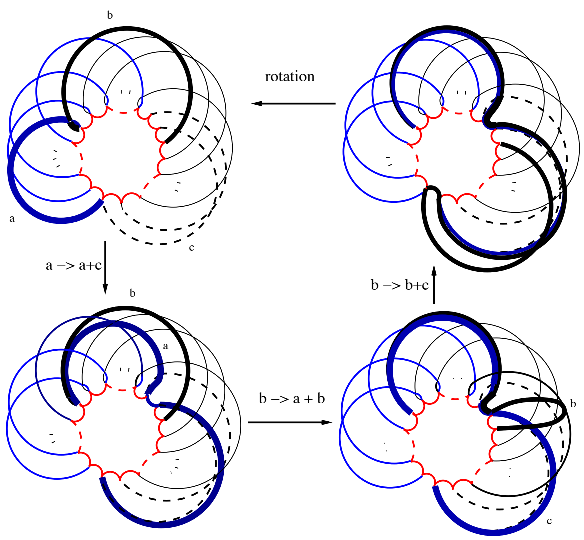

Figure 3: Train track maps for ϕ_n.

The corresponding digraphs are shown in Figure 4.

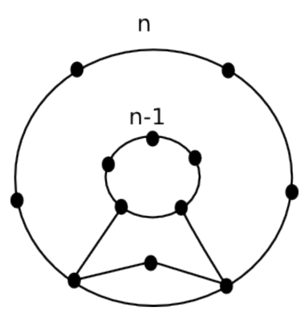

Figure 4: Digraphs associated to ϕ_n.

References

[McM] C. McMullen. Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations. *Ann. Sci. École Norm. Sup.* **33** (2000), 519–560.