The dynamics of mapping classes on surfaces

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9 Penner examples and quasi-periodicity.

In this lecture, we will study a sequence of pseudo-Anosov mapping classes defined by R. Penner [Pen], and describe how they correspond to points on a fibered face.

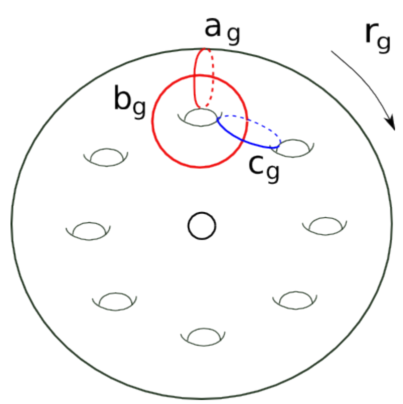

Figure 1: Penner's sequence

Consider the family of mapping classes $\phi_g : S_g \to S_g$ shown in Figure 1. Here S_g is a compact oriented surface of genus g with two boundary components, and ϕ_g is the composition

$$\phi_g = r_g \circ d_{c_g} \circ d_{b_g}^{-1} \circ d_{a_g}.$$

Lemma 1 (Penner)

$$\lambda(\phi_g)^g \leq 11.$$

Penner constructed this example and proved Lemma 1 as a key step in proving that the minimum dilatations δ_g for pseudo-Anosov mapping classes on closed genus g surfaces behaves asymptotically like

$$\log(\delta_g) \asymp \frac{1}{g}.$$

Penner's Lemma can be generalized to other similar examples (see [Bau], [Val]). Let (S, α) be a pair with Σ a compact oriented surface with boundary, and α a relatively closed curve on S. Let $\Sigma = S \setminus \alpha$ be the closure where α is replaced by two copies α^+ and α^-. Now construct surfaces Y_n by gluing together Σ in an circular chain as in Figure 2

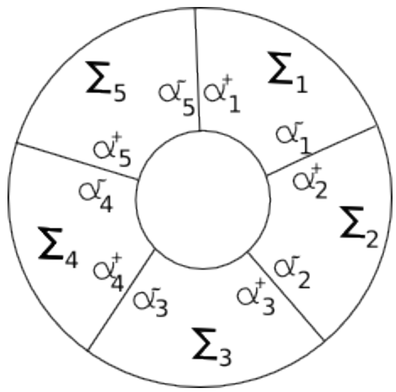

Figure 2: Circular chain of surfaces

The surfaces Y_n are n-cyclic coverings of S,

$$\rho_n : Y_n \to S,$$

and have covering automorphism group $\mathbb{Z}/n\mathbb{Z} = \langle r_n \rangle$. Fix $k > 0$, and assume $n \geq k$. Let γ be a relative multi-curve on S whose algebraic intersection with α is zero, and let γ_n be a lift to Y_n that is contained in $\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k$, and is compatibly chosen for all n. Let d_n be the Dehn twist defined on γ_n.

We call (S, α) a *wedge* and γ a *connecting curve*. For any mapping class $\eta : \Sigma \to \Sigma$, let $\eta_n : Y_n \to Y_n$ be the mapping class defined by

$$\eta_n(x) = \begin{cases} \eta(x) & \text{if } x \in \Sigma_1 \\ x & \text{otherwise} \end{cases}$$

Theorem 2 ([Hir]) *Let (S, α) be a wedge, γ a connecting curve. Let $\eta : \Sigma \to \Sigma$ be any mapping class such that $d_\gamma \circ \eta$ is pseudo-Anosov. Then*

1. *$f_n = r_n \circ d_n \circ \eta_n$ is pseudo-Anosov for each n, and*
2. *there is a constant C independent from n such that*

$$\lambda(\phi_n)^{|\chi(Y_n)|} \leq C.$$

Outline of Proof.

Let $\phi : S \to S$ be defined by $\phi = d_\gamma \circ \eta$.

Step 1. The mapping class triple $(S, d_\gamma \circ \eta, \alpha)$ defines a linear section of the fibered face associated to $(S, d_\gamma \circ \eta)$.

Step 2. Let $\rho : \widetilde{S} \to S$ be the cyclic covering of S defined by α. Let $\zeta : \widetilde{S} \to \widetilde{S}$ be the group of covering automorphisms, and let $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ be the lift of ϕ.

Then ζ and $\widetilde{\phi}$ determine commuting automorphisms

$$\begin{aligned} Z : \widetilde{S} \times \mathbb{R} &\to \widetilde{S} \times \mathbb{R} \\ (x,t) &\mapsto (\zeta(x), t) \end{aligned}$$

and

$$\begin{aligned} T : \widetilde{S} \times \mathbb{R} &\to \widetilde{S} \times \mathbb{R} \\ (x,t) &\mapsto (\widetilde{\phi}(x), t-1). \end{aligned}$$

Then Z and T generate $\mathrm{Aut}_M(\widetilde{M})$.

Step 3. Let $\xi \in H^1(M;\mathbb{Z})$ be the dual to Z, and let

$$\psi_n = n\psi + \xi.$$

By Thurston's theory of fibered faces, we have, for large n, ψ_n defines a fibration of M. Let $\phi_n : S_n \to S_n$ be the corresponding monodromy. Then we have

1. ϕ_n is pseudo-Anosov, and
2. the normalized dilatations $\overline{\lambda}(\phi_n)$ converge to $\overline{\lambda}(\phi)$.

Step 4. We have left to show that (Y_n, f_n) and (S_n, ϕ_n) agree. To do this we need a lemma.

Choose a lift of Σ in $\widetilde{S}$ and call it Σ_0. Let $\eta_i : \widetilde{S} \to \widetilde{S}$ be the the result of lifting η to $\Sigma_i = \zeta^i(\Sigma_0)$ and extending by the identity to its complement in $\widetilde{S}$. Let $\widetilde{d} : \widetilde{S} \to \widetilde{S}$ be the Dehn twist on $\widetilde{S}$ centered at the lift of γ to

$$\widetilde{\gamma} \subset \zeta^1(\Sigma_0) \cup \cdots \cup \zeta^k(\Sigma_0).$$

By construction, there is one so that $\rho_n : \widetilde{S} \to Y_n$ equals γ_n for all n.

Lemma 3 *For $x \in \Sigma_0$, and $n \geq k$,*

$$\widetilde{\phi}(x) = \zeta^{-n}(\zeta \circ d_{\widetilde{\gamma}} \circ \widetilde{\eta}_0)^n.$$

Let

$$\begin{aligned} \widehat{T}_n : \widetilde{S} \times \mathbb{R} &\to \widetilde{S} \times \mathbb{R} \\ (x,t) &\mapsto (\widehat{(}x), t - \frac{1}{n}). \end{aligned}$$

where

$$\widehat{\phi} = d_{\widetilde{\gamma}} \circ \eta_0.$$

The maps Z and T_n do not commute. Let $R = Z \circ T_n$.

Define $\widetilde{S}_n \subset \widetilde{S} \times \mathbb{R}$ to be the surface

$$\widetilde{S}_n = \left(\bigcup_{i \in \mathbb{Z}} R^i(\Sigma_0) \right) \cup \left(\bigcup_{i \in \mathbb{Z}} R^i(\alpha^-) \times [\frac{i}{n}, \frac{i+1}{n}] \right).$$

(To truly make $\widetilde{S}_n$ transverse to the flow defined by ϕ, it suffices to tilt the suspensions of α^- slightly.) The embedding of $\widetilde{S}_n$ in M^{ab} is illustrated in Figure 3.

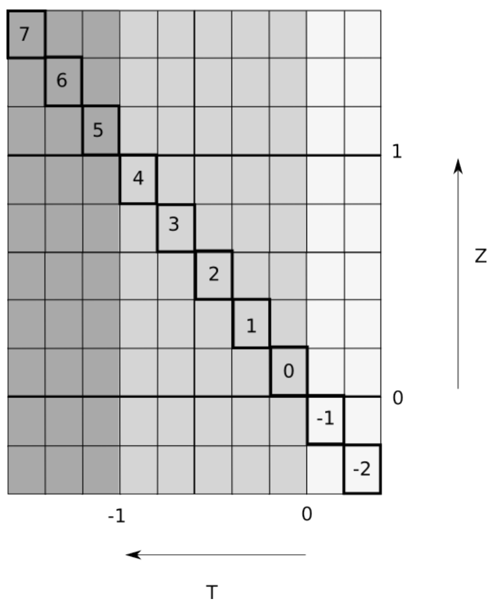

Figure 3: A picture of $\widetilde{S}_n$ in $\widetilde{M}^{\text{ab}} = \widetilde{S} \times \mathbb{R}$.

Lemma 4 *$\widetilde{S}_n$ is a connected subsurface of $\widetilde{S} \times \mathbb{R}$.*

Proof. Since R is the identity on $\alpha^- \subset \Sigma_0$, we have

$$R^i(\alpha^-) = R^{i-1}(\alpha^+).$$

□

Lemma 5 *$\widetilde{S}_n$ is invariant under the action of R, and its quotient by R^n is homeomorphic to Y_n.*

By Lemma 3, $R^n = TZ^n$, and hence it generates the kernel of ψ_n. Thus we have

Corollary 6 $S_n = Y_n$.

To find $\widetilde{\phi}_n$, where ϕ_n is the monodromy of ψ_n, we note that Z^{-1} satisfies

$$\psi_n(Z^{-1}) = -1.$$

Thus,

$$Z^{-1} = T_{\widetilde{\phi_n}}.$$

Using the local product structure of $\widetilde{S} \times \mathbb{R}$, we see that if one takes an $x \in \widetilde{S}_n$ applies Z^{-1} and then flows back to $\widetilde{S}_n$ using the product structure, then the result is

$$x \mapsto \widehat{\phi}^{-1}(R^{-1}(x)).$$

It follows that $\widetilde{\phi}_n = R \circ \widehat{\phi}$, which is a lift of f_n. $\square$

Exercise Consider the Penner example.

1. Show that the mapping torus has $b_1 = 2$.
2. Show that the Teichmüller polynomial for the Penner example is
$$u^2 - u(5 + t + t^{-1}) + 1.$$
3. Show that $\lambda(\phi_g)$ is the largest root of
$$x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$
4. Show that $\overline{\lambda}(\phi_g)$ converges to
$$\left(\frac{7 + 3\sqrt{5}}{2}\right)^2.$$

References

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[Val] A. Valdivia. Asymptotics of minimal dilatation pseudo-Anosov mapping classes on rays in the gn-plane. *Florida State Univeristy, PhD. Thesis* (2011).