ıen doing induction on the number of vertices or edges of a graph, we have said that you :t with $k+1$ vertices or edges and *remove* one. It is not equivalent to start with k edges or tices and *add* one. Why is this so?

u have seen that proving that any map can be colored with six colors is straightforward. It ıot too much more difficult to prove that all maps can be colored with five. It has long been *piricially* known by cartographers that *four* colors suffice, but the mathematical question ʼery difficult and went unsolved for more than one hundred years.

The four color map problem was reduced to checking more than a thousand possible .es. The question was then "settled" by having a computer check all the cases. Most thematicians consider the issue resolved, but there is nonetheless some uneasiness about solution. Historically, mathematical questions have been considered to be solved when ›erts in the field read the proof and declare it to be sound. The difficulty is that when a nputer checks cases and comes forward with an answer, there is no proof to publish and :ck. Experts can check the computer program that produced the answer, of course, and this s what was done in the case of the four color map theorem. On the other hand, this brings a host of philosophical questions.

)o the experts that check the program have to understand all the workings of the machine ıat was used to run it? What if there is a problem with the machine? (If you think this is ar-fetched, don't forget that the first commercial version of the Pentium computer chip did ırdinary division incorrectly!)

n the case of the four color theorem, the cases were determined by humans and checked by ı machine. What will the mathematical community do when the machine not only checks he cases but also determines what they are in the first place? (Can such a time be very far ıway?)

e there other questions that arise in your mind? What, in your opinion, would be a reasonable urse for the mathematical community with respect to computer-generated proofs?[7]

philosophical discussion along other lines, see *Pi in the Sky* by John D. Barrow (Oxford University 1992.)

5 Functions

5.1 Basic Ideas

In your previous studies you have, no doubt, run across the concept of a function. In fact, you have probably spent a large percentage of your mathematical education considering functions in one context or another. They are in every mathematician's tool chest; we use them all the time. This chapter will formalize the idea of a function and begin to establish some of the accompanying standard ideas and techniques available to us.

You probably first encountered functions in the context of a function from $\mathbb{R}$ to $\mathbb{R}$, that is, a correspondence that takes each real number and assigns to it a real number. There are countless possibilities. The function $f(x) = 3x - 1$ assigns to a given input number the number obtained by multiplying the input by 3 and subtracting 1 from the result. Another function may interpret a given number as a fixed time and assign to it the value of a particular stock at that time. Of course, we need not restrict ourselves to thinking of functions in which the inputs and outputs are numbers. A table in which the first column lists the names of a bunch of dairy cows and the second column indicates the average daily production of milk by each cow is an example of a function.

Name	*Milk Produced per Day (in pounds)*	*Name*	*Milk Produced per Day (in pounds)*
Alice	39.5	Kicker	37
Big Mama	43	Sassy	38.5
Bluebell	35	Spots	38
Clara	39.5	Sue Ellen	39
Jane	40.5	Tilly	39

suppose we have a collection of subsets of $\mathbb{N}$. We could assign to each the subset of hat we get by adding 1 to each element of the original set. For instance,

$$\{1, 5, 10\} \to \{2, 6, 11\}.$$

eneral, a function from a set A to a set B is a correspondence by which an element of ; assigned to every element of A. The essential ingredient here is that *each* element l have a *single* element of B assigned to it.

In practice, people usually think of functions in this informal way. It conjures up /namic picture of something that takes elements of A and *transforms* them into ıents of B. Unfortunately, it is not mathematically rigorous. Just what exactly is a respondence"? What does it mean to "assign" an element of B to a given element of Well, first of all, we are considering *pairs* of elements—each element of A is paired an element of B. That is, a function is a relation. However, not just any relation ch of ordered pairs) will do. We said that the key ingredient in our discussion was specification that each element of A has a single element of B assigned to it. The wing definition lays this out formally.

1 DEFINITION

4 and B be nonempty sets. A **function** f from set A to set B (denoted by $f : A \to B$) elation between A and B satisfying the following conditions:

1. For each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$, and
2. if (a, b) and (a, c) are in f, then $b = c$.

$\in A$, the unique element $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$.

ark. Using the familiar language of functions, this says that if $a \in A$, we never to justify the existence of $f(a)$; it is always there. Furthermore, a maps to only one . If we know that a maps to b and a also maps to c, then c and b must be equal.

2 EXERCISE

$$A = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\},$$
$$B = \{\flat, \sharp, \natural\},$$
$$C = \{\emptyset, \exists, \forall, \infty, \otimes\}.$$

h of the following relations are functions from one of the above sets to another? in.

. $\{(\flat, \heartsuit), (\sharp, \clubsuit), (\natural, \spadesuit)\}$
. $\{(\emptyset, \diamondsuit), (\exists, \spadesuit), (\otimes, \heartsuit), (\infty, \heartsuit)\}$
. $\{(\flat, \heartsuit), (\sharp, \heartsuit), (\natural, \heartsuit)\}$
. $\{(\diamondsuit, \flat), (\clubsuit, \flat), (\heartsuit, \sharp), (\spadesuit, \natural)\}$

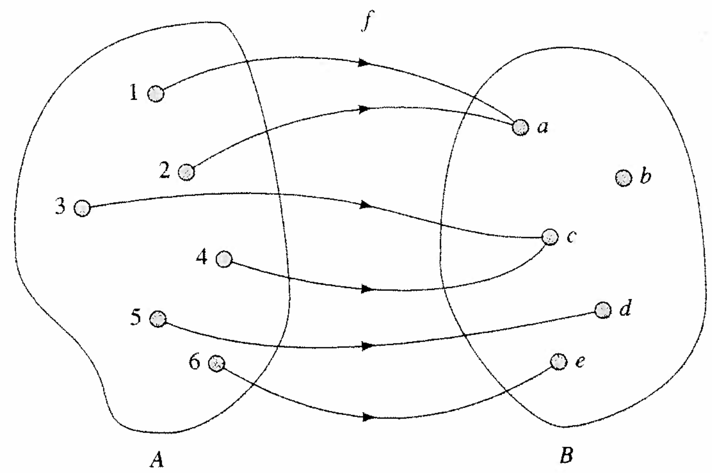


Figure 5.1 This is a useful way to represent a function between two finite sets A and B.

5. $\{(\diamondsuit, \heartsuit), (\clubsuit, \diamondsuit), (\heartsuit, \diamondsuit), (\spadesuit, \heartsuit)\}$
6. $\{(\exists, \diamondsuit), (\emptyset, \clubsuit), (\forall, \heartsuit), (\exists, \heartsuit), (\infty, \diamondsuit), (\otimes, \clubsuit)\}$

Choose one set that you decided was a function. Draw and carefully label a graph of this function as illustrated in Figure 5.1. □

5.1.3 EXERCISE

Consider the function shown in Figure 5.1. What is $f(1)$? What is $f(2)$? What is $f(5)$? □

5.1.4 EXERCISE

When discussing functions from $\mathbb{R}$ to $\mathbb{R}$, many high school teachers tell their students that they can tell from a graph whether they have a function by determining whether the graph passes "the vertical line test"—that is, whether each vertical line in the plane crosses the graph exactly once. Use the language of ordered pairs to explain why this works. □

5.1.5 DEFINITION

In Definition 5.1.1, the set A is called the **domain** of the function f. It is denoted by $\mathcal{D}om(f)$.

The set B is called the **codomain** of the function f. It is denoted by $\mathcal{C}odom(f)$.

The set $\mathcal{R}an(f) = \{b \in B : \text{there exists } a \in A \text{ such that } b = f(a)\}$ is called the **range** of the function f or the **image of** A under the function f.

l.6 EXERCISE

ısider the function shown in Figure 5.1. What are the domain, codomain, and range this function? □

Two functions are equal if they have the same domain, the same codomain, and ree" on every element of the domain. More formally,

.7 THEOREM (Equality of functions)

$\cdot : A \to B$ and $g : A \to B$ are functions, then $f = g$ if and only if $f(a) = g(a)$ for $\imath \in A$.

(*Hint:* A function is a relation. Saying that two functions are equal is to say they are tions between the same pair of sets, and they include the same ordered pairs.) □

The tool by which we as mathematicians compare mathematical structures is the ction. To make the comparisons meaningful, it is usually desirable to consider func-s with special properties. For instance, in the study of calculus, one looks at contin-s functions and differentiable functions. In many mathematical contexts it is useful onsider functions that are *one-to-one* and functions that are *onto*.

To get the general idea of why we are interested in these properties, recall our short ussion of isomorphic partial orders on page 73. We said that if two partial orderings e "the same up to a relabeling of the elements," then they were to be called isomorphic ial orders. Of course, we have a good intuitive idea of what it means in that context elabel elements. Each element in the first partial order can be "associated" with an nent in the second partial order in a natural way. This is simply a function from the set of labels to the second set of labels! However, the word "relabel" has some ific connotations. First of all, we will require that there are enough new names to around"—that is, that we won't have to use the same name twice. Each element le second list can be assigned *only once*. Furthermore, since we are comparing two erent structures, we want to account for all the elements in each—we don't want to any new names left over.

.8 DEFINITION

ınction $f : A \to B$ is said to be **one-to-one** if given $b \in B$, there is at most one $a \in A$ which $b = f(a)$.

A function $f : A \to B$ is said to be **onto** if for each $b \in B$, there is at least one $a \in A$ which $b = f(a)$. In other words, f is onto if the codomain and the range of f are the e set.

A function that is both one-to-one and onto is often called a **one-to-one correspon-ce.**[1]

---

er common usage: One-to-one functions are sometimes called **injective**. Onto functions are sometimes l **surjective**. In this spirit, one-to-one correspondences are called **bijective**. Interestingly, many mathe-ians mix the usage and say one-to-one, onto, and bijective.

5.1.9 EXERCISE

In the context of relabeling, which of the conditions above corresponds to "having enough new names to go around"? Which corresponds to "not having any new names left over"? □

5.1.10 EXERCISE

1. Give an example of finite sets A and B, and a function $f : A \to B$ in which f is one-to-one but not onto.
2. Give an example of finite sets A and B, and a function $f : A \to B$ in which f is onto but not one-to-one.
3. Give an example of finite sets A and B, and a function $f : A \to B$ in which f is both one-to-one and onto.
4. Give an example of finite sets A and B, and a function $f : A \to B$ in which f is neither one-to-one nor onto.

Pictures such as Figure 5.1 will help you to visualize the concepts of one-to-one and onto for functions between finite sets. □

5.1.11 EXERCISE

Thinking some more about "the vertical line test," we can ask about a possible "horizontal line test." Such tests can be used to determine whether the graph of a function from $\mathbb{R}$ to $\mathbb{R}$ depicts a function that is one-to-one, onto, neither, or both. Devise a horizontal line test and use the language of ordered pairs to explain how it works. □

5.1.12 EXERCISE

1. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ in which f is one-to-one but not onto.
2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ in which f is onto but not one-to-one.
3. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ in which f is both one-to-one and onto.
4. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ in which f is neither one-to-one nor onto.

Illustrate your examples by drawing the graphs of the functions that you chose. □

5.1.13 THEOREM

Let $f : A \to B$ be a function. The following conditions on f are equivalent:

i. f is one-to-one.

ii. For all a_1 and a_2 in A, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

(*Hint:* Try proof by contraposition.) □

Showing a function is one-to-one: Remember that equivalent statements are just different ways of saying the same thing. Thus the second statement in Theorem 5.1.13 is *another way of defining what it means for a function to be one-to-one.* When attempting to prove that a function $f : A \to B$ is one-to-one, we almost always use the alternative phrasing rather than Definition 5.1.8. This gives us a straightforward procedure for proving that a function is one-to-one:

> *We begin by assuming that* $f(a_1) = f(a_2)$ *and then show that* a_1 *must equal* a_2.

(Notice that this is very much like the procedure for showing that something is unique. Explain why this makes sense.)

.14 EXERCISE

each of the functions $f : \mathbb{R} \to \mathbb{R}$ given below either show that f is one-to-one or ve that it is not.

1. $f(x) = \frac{x}{2} + 6$ 2. $f(x) = 4x^3$ 3. $f(x) = x^3 - x$
4. $f(x) = e^x$ 5. $f(x) = \sin(x)$ □

Showing that a function is onto: Stating that a function f is onto is to state an existence theorem: Given an arbitrary element b in the codomain of f, *there exists* an element a in the domain of f such that $f(a) = b$. (See page 28 for a review of the procedure for proving existence theorems.)

l.15 EXERCISE

ermine whether the functions given in Exercise 5.1.14 are onto. Prove your answers. □

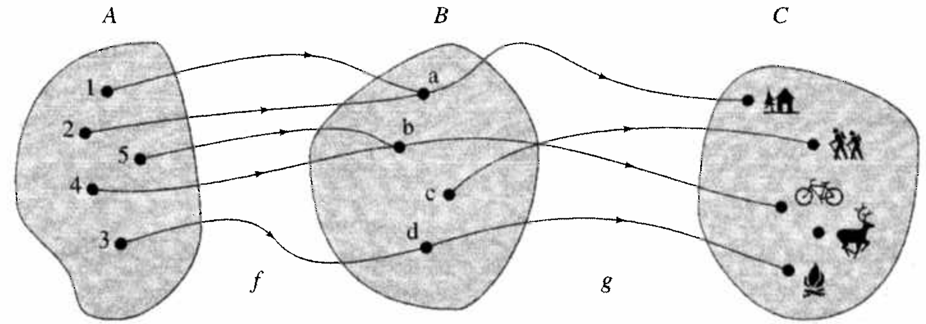
We finish the section with a theorem that will be used in Chapter 7.

l.16 THEOREM

t X and Y be sets. If $f : X \to Y$ is a function, then there is an onto function $f^* : X \to dom(f)$ such that for all $x \in X$, $f(x) = f^*(x)$.

In addition, if f is one-to-one, f^* is a one-to-one correspondence. □

What Theorem 5.1.16 tells us is that if we have a function that is not onto, we can : our hands on a closely related function that *is* onto by changing our point of view ıit—by thinking of the same set of pairs as a function with the same domain and a ferent (smaller) codomain.



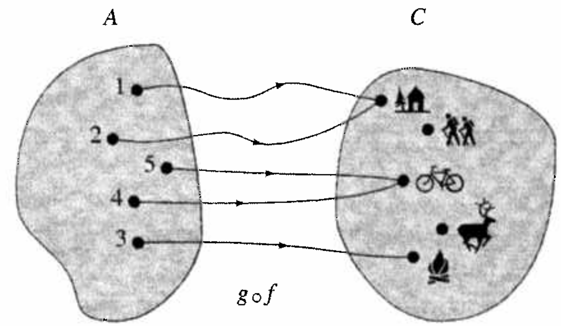


Figure 5.2 Composition of functions

5.2 Composition and Inverses

5.2.1 DEFINITION

If $f : A \to B$ and $g : B \to C$ are functions, then a new function $g \circ f : A \to C$ can be defined by $(g \circ f)(a) = g(f(a))$. This new function is called the **composition** of g and f. (See Figure 5.2.)

5.2.2 EXERCISE

1. Give an example of finite sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ for which f is onto, but $g \circ f$ is not.
2. Give an example of finite sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ for which g is onto, but $g \circ f$ is not.
3. Give an example of finite sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ for which f is one-to-one, but $g \circ f$ is not.
4. Give an example of finite sets A, B, and C, and functions $f : A \to B$ and $g : B \to C$ for which g is one-to-one, but $g \circ f$ is not. □

2.3 THEOREM

ppose that $f : A \to B$ and $g : B \to C$ are functions. Then the following hold.

1. If f and g are both one-to-one, $g \circ f$ is one-to-one.
2. If f and g are both onto, $g \circ f$ is onto. □

2.4 PROBLEM

ppose that $f : A \to B$ and $g : B \to C$ are functions. In each of the following cases, swer the question. If your answer is "yes," give a proof. If your answer is "no," give ounterexample and say what additional hypotheses are needed to make the statement e; then prove the statement with the additional hypotheses.

1. If $g \circ f$ is one-to-one, must f be one-to-one?
2. If $g \circ f$ is one-to-one, must g be one-to-one?
3. If $g \circ f$ is onto, must f be onto?
4. If $g \circ f$ is onto, must g be onto? □

2.5 THEOREM

mposition of functions is associative. That is, if $f : A \to B$, $g : B \to C$, and $h : C \to D$ functions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

roblem 5.2.6 is a guide for proving this theorem.) □

2.6 PROBLEM

is problem relates to the proof of Theorem 5.2.5. In proving that theorem, you should rt, of course, by verifying that

$$h \circ (g \circ f) \quad \text{and} \quad (h \circ g) \circ f$$

ve the same domain and codomain. (Do that now!)

Then it is necessary to show that for all $a \in A$,

$$(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a).$$

the table on the next page you will find four arguments purporting to show this fact. ly one is completely correct. The others vary in how far they stray. Critique each gument, describing as many errors as you find. Finally, identify the correct argument.

(*Note:* The arguments may fall short for a variety of reasons. They may be ill-nceived, starting and ending in the wrong places; they may contain lapses in logic;[2] they may apply the notion of function composition incorrectly.)

emember the caution made in the introductory essay: "An argument consisting entirely of true statements ot valid if the connecting inferences are not justified by logic."

Argument 1	*Argument 2*
$(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ $h \circ (g(f(a))) = (h \circ g) \circ f(a)$ $h(g(f(a))) = h(g(f(a)))$	$(h \circ (g \circ f))(a) = h((g \circ f)(a))$ $= h(g(f(a)))$ $= (h \circ g)(f(a))$ $= ((h \circ g) \circ f)(a)$

Argument 3	*Argument 4*
$(h \circ (g \circ f))(a) = h((g \circ f)(a))$ $= h(g(f(a)))$ and $((h \circ g) \circ f)(a) = h \circ (g(f(a)))$ $= h(g(f(a)))$	$(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ $h((g \circ f)(a)) = (h \circ g)(f(a))$ $h(g(f(a))) = h(g(f(a)))$

□

5.2.7 THEOREM

Let $f : A \to B$ be a function. The relation

$$I = \{(f(a), a) : a \in A\}$$

is a function if and only if f is a one-to-one correspondence. □

5.2.8 DEFINITION

Let f be a one-to-one correspondence. The function defined in Theorem 5.2.7 is called the **inverse** of the function f. It is denoted by f^{-1}.

5.2.9 THEOREM

Let $f : A \to B$ be a function.

Part I. Then if f is a one-to-one correspondence, the following statements hold.

1. $(f \circ f^{-1})(x) = x$ for all $x \in B$.
2. $(f^{-1} \circ f)(x) = x$ for all $x \in A$.
3. If $g : B \to A$ is any function for which
   - $(f \circ g)(x) = x$ for all $x \in B$, and
   - $(g \circ f)(x) = x$ for all $x \in A$,

   then $g = f^{-1}$.

(In other words, f^{-1} satisfies these two composition conditions, and, furthermore, it is the only function that does.)

Part II. Conversely, if f is any function from A to B and there exists a function $g : B \to A$ satisfying $(f \circ g)(x) = x$ for all $x \in B$ and $(g \circ f)(x) = x$ for all $x \in A$, then f is a one-to-one correspondence. (Thus f^{-1} exists and must, in turn, be equal to g by Part I of the theorem.) □

This theorem allows us easily to prove the relationship between the inverse of a composition of two functions and the inverses of the functions being composed.

5.2.10 THEOREM

Suppose that $h : A \to B$ and $k : B \to C$ are functions. If h and k are both one-to-one and onto, $(k \circ h)^{-1}$ is a function and $(k \circ h)^{-1} = h^{-1} \circ k^{-1}$. □

5.3 Images and Inverse Images

Theorems 5.2.7 and 5.2.9 show that unless f is one-to-one and onto, there is no meaningful way to define an inverse function for f. It is often useful to consider a more general concept which is closely related to that of the inverse function—the inverse image.

5.3.1 DEFINITION

Let $f : A \to B$ be a function. Let $S \subseteq B$. The set

$$f^{-1}(S) = \{a \in A : f(a) \in S\}$$

is called the **inverse image** of the set S under the function f.

5.3.2 EXAMPLE

Consider the function illustrated in Figure 5.3. What is $f^{-1}(S)$? ■

5.3.3 EXERCISE (Say it in words)

Let $f : A \to B$ be a function. Let $X \subseteq B$. Let $z \in A$. Complete the following sentence:

$$z \in f^{-1}(X) \text{ means that } f(z) \_\_\_\_\_\_\_\_\_\_\_\_ .$$ □

We have said that the inverse of a function and inverse image of a set under a function are closely related concepts; thus we use similar notation to denote them. Be careful, though. They are not the *same* concept! The inverse of a function is a *function,* and we have seen that this exists only if the function is one-to-one and onto. The inverse image of a set under a function is a *set* and exists for all functions, not just those that are one-to-one correspondences. Get straight both the differences and the similarities, and when you see the symbol f^{-1} be sure that you understand whether it denotes a function or a set. (It should be clear from the context.) Keep the distinction in mind when you do the remaining exercises in this section.

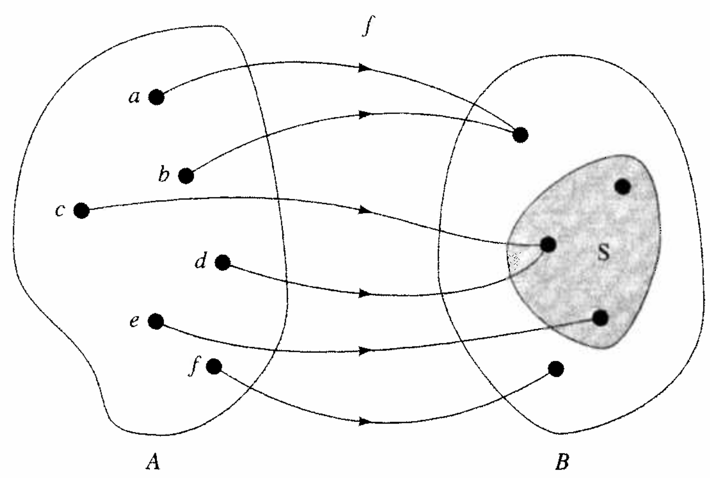


Figure 5.3 What is the inverse image of S?

5.3.4 EXERCISE

Give an example of a function $f : A \to B$ and a nonempty subset S of B for which $f^{-1}(S) = \emptyset$. □

5.3.5 EXERCISE

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Find the following.

1. $f^{-1}(\{4\})$
2. $f^{-1}([-2, 9])$
3. $f^{-1}((1, 4])$ □

The following theorem gives some useful properties of inverse images.

5.3.6 THEOREM

Let $f : A \to B$ be a function. Let Λ be an arbitrary indexing set. Let $\{S_\alpha\}_{\alpha \in \Lambda}$ be a collection of subsets of B, and let S be any subset of B. Then

1. $f^{-1}(\bigcup_{\alpha \in \Lambda} S_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(S_\alpha)$.
2. $f^{-1}(\bigcap_{\alpha \in \Lambda} S_\alpha) = \bigcap_{\alpha \in \Lambda} f^{-1}(S_\alpha)$.
3. $f^{-1}(S^{\complement}) = (f^{-1}(S))^{\complement}$.

(*Hint:* Since these are sets, you will need to employ element arguments.) □

In keeping with the spirit and notation of Definition 5.3.1, we define the image of a subset of A under the function f.