UV^k-Mappings

John Bryant, Steve Ferry, and Washington Mio

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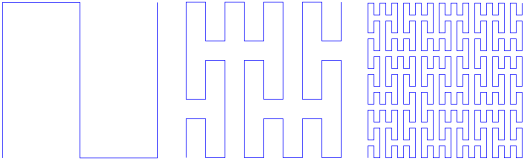
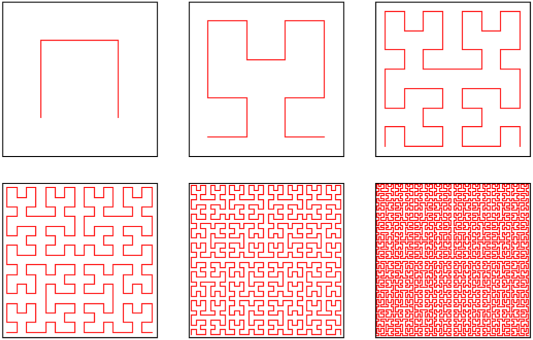
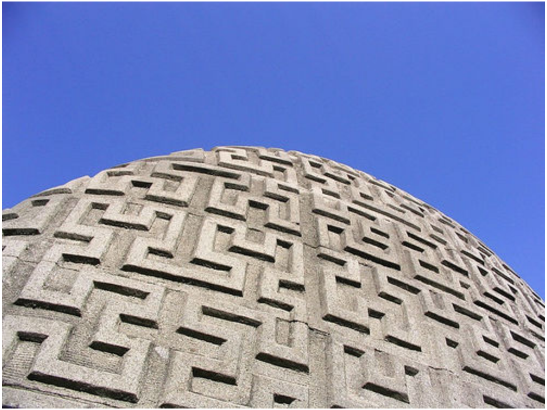
Continuous maps can raise dimension:
Peano curve (Giuseppe Peano, ca. 1890)

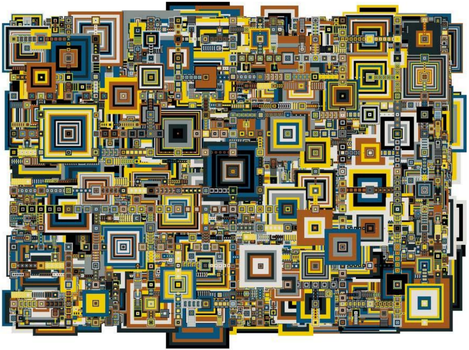


Hilbert Curve ($\sim$ 1890)





LA CITTA' DI CUNEO
NELL'VIII CENTENARIO
DELLA SUA FONDAZIONE
ALLO SCIENZIATO
GIUSEPPE PEANO
"CURVA DI PEANO"
OPERA DI
DARIO GHIBAUDO
1998



Lyudmila Keldysh (1957): There is an *open*, surjective map $f\colon I^3 \to I^4$ with connected point-inverses, pseudoisotopic to the inclusion map.

A. V. Černavskii (1985): If $2k + 3 \leq n$, there are surjections

$$f\colon I^n \to I^{n+p}$$

$$g\colon S^n \to S^{n+p}$$

with k-connected point-inverses, pseudoisotopic to the inclusion map.
In particular, if $n \geq 5$, there is a surjection

$$g\colon S^n \to S^{n+1}$$

with simply connected point-inverses.
How might these be useful?
A digression:

M^n closed topological n-manifold

Structure set

$\mathcal{S}(M)$ = set of homotopy equivalences $g\colon N \to M$,
N a topological n-manifold, modulo relation
$g_1 \sim g_2$ if there is a homeomorphism
$h\colon N_1 \to N_2$
making the diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{\;h\;} & N_2 \\ & {\scriptstyle g_1}\searrow \quad \swarrow{\scriptstyle g_2} & \\ & M & \end{array}$$

homotopy commutative.

If M is simply connected there is a surgery exact sequence

$$\cdots \to [(M \times I, \partial), G/TOP] \to L_{n+1} \to \mathcal{S}(M) \to [M, G/TOP] \to L_n$$

where $\mathbb{L}$ is the simply connected surgery spectrum,
$L_k = \mathbb{Z}, 0, \mathbb{Z}/2\mathbb{Z}, 0$
accordingly as
$k \equiv 0, 1, 2, 3 \bmod 4$,
and
$[M, G/TOP]$ is the set of normal bordism classes of degree one normal maps (normal invariants or "surgery problems over M")
$g \colon N \to M$,
where N is a topological n-manifold.

Given a “control” map

$p\colon M \to B$ (metric space)

there is a “controlled” structure set

$$\mathcal{S}_\epsilon\begin{pmatrix} M \\ \downarrow p \\ B \end{pmatrix} = \text{set of homotopy equivalences}$$

$g\colon N \to M$,

where

$g_1 \sim g_2$ if there is a homeomorphism

$h\colon N_1 \to N_2$

making the diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{\;h\;} & N_2 \\ & {\scriptstyle g_1}\searrow \quad \swarrow{\scriptstyle g_2} & \\ & M & \\ & \downarrow p & \\ & B & \end{array}$$

ϵ-homotopy commutative over B, where $\epsilon \leq \epsilon_0$ a “critical ϵ,” which depends on B.

If M is simply connected and $p\colon M \to B$ is a surjection with simply connected point inverses (so B is also simply connected), there is a controlled surgery exact sequence

$$\cdots \to H_{n+1}(B;\mathbb{L}) \to \mathcal{S}_\epsilon\begin{pmatrix} M \\ \downarrow p \\ B \end{pmatrix} \to [M, G/TOP] \to H_n(B;\mathbb{L})$$

(Ferry, Pedersen-Quinn-Ranicki)

Sequences are functorially related by the constant map $B \to pt$

$$
\begin{array}{ccccccc}
\cdots \to H_{n+1}(B;\mathbb{L}) & \to & \mathcal{S}_\epsilon\begin{pmatrix} M \\ \downarrow^{p} \\ B \end{pmatrix} & \to & [M, G/TOP] & \to & H_n(B;\mathbb{L}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to L_{n+1} & \to & \mathcal{S}(M) & \to & [M, G/TOP] & \to & L_n \\
|| & & || & & & & || \\
H_{n+1}(pt;\mathbb{L}) & & \mathcal{S}_\epsilon\begin{pmatrix} M \\ \downarrow^{p} \\ pt \end{pmatrix} & & & & H_n(pt;\mathbb{L})
\end{array}
$$

Apply to a "Černavskii map" $p\colon S^n \to S^{n+1}$ $(n \geq 5)$

$$
\begin{array}{ccccccc}
\cdots \to H_{n+1}(S^{n+1};\mathbb{L}) & \to & \mathcal{S}_\epsilon\begin{pmatrix} S^n \\ \big\downarrow p \\ S^{n+1}\end{pmatrix} & \xrightarrow{0} & [S^n, G/TOP] & \to & H_n(S^{n+1};\mathbb{L}) \\
\downarrow & & \downarrow & & \downarrow\approx & & \downarrow \\
\cdots \to L_{n+1} & \to & \mathcal{S}(S^n) & \to & [S^n, G/TOP] & \to & L_n \\
 & & \| & & & & \\
 & & 0 & & & &
\end{array}
$$

By the Poincaré conjecture (and *h*-cobordism thm)

$$\mathcal{S}(S^n) = 0 (= \mathcal{S}(S^n \times I, \partial))$$

$$H_{n+1}(S^{n+1}; \mathbb{L}) \cong \bigoplus_{p+q=n+1} H_p(S^{n+1}; L_q)$$

$$\cong H_{n+1}(S^{n+1}; \mathbb{Z}) \oplus H_0(S^{n+1}; L_{n+1})$$

$$\cong \mathbb{Z} \oplus L_{n+1}$$

$$
\begin{array}{ccccccccc}
\mathcal{S}_\epsilon\begin{pmatrix} (S^n \times I, \partial) \\ \big\downarrow p \\ S^{n+1} \end{pmatrix} & \xrightarrow{0} & [(S^n \times I, \partial), G/TOP] & \rightarrow & \mathbb{Z} \oplus L_{n+1} & \rightarrow & \mathcal{S}_\epsilon\begin{pmatrix} S^n \\ \big\downarrow p \\ S^{n+1} \end{pmatrix} & \xrightarrow{0} \\
\downarrow & & \downarrow\approx & & \downarrow proj & & \downarrow 0 & \\
0 & \rightarrow & [(S^n \times I, \partial), G/TOP] & \xrightarrow{\approx} & L_{n+1} & \xrightarrow{0} & \mathcal{S}(S^n) & \rightarrow
\end{array}
$$

$$
\Longrightarrow \quad \mathcal{S}_\epsilon\begin{pmatrix} S^n \\ \big\downarrow p \\ S^{n+1} \end{pmatrix} \cong \mathbb{Z}
$$

Given a class corresponding to a nontrivial element of

$$\mathcal{S}_\epsilon\left(\begin{array}{c} S^n \\ \downarrow p \\ S^{n+1} \end{array}\right)$$

and $\delta > 0$, then for every $\mu > 0$, there is representative

$$g\colon S^n \to S^n$$

such that

g is a μ-equivalence over S^{n+1},

g is δ-homotopic to a ν-equivalence over S^{n+1} for any $\nu > 0$, and

g is not ϵ-homotopic to a homeomorphism.

Maps such as these were used as gluing maps by B-F-M-Weinberger in the construction of non-resolvable homology manifolds.

In fact, the integer ι representing an element of $\mathcal{S}_\epsilon\begin{pmatrix} S^n \\ \downarrow p \\ S^{n+1} \end{pmatrix}$ corresponds to a homology $(n+1)$-manifold X, homotopy equivalent to S^{n+1}, having Quinn's resolution obstruction $1 + 8\iota \in 1 + 8\mathbb{Z}$.

A compact metric space C has property $\mathbf{UV^k}$, $k \geq 0$, if, for some (hence, any) embedding $C \subseteq X$, an ANR, every nbd U of C contains a nbd V of C such that

$$\pi_i(V) \to \pi_i(U)$$

is the zero-homorphism for
$0 \leq i \leq k$.

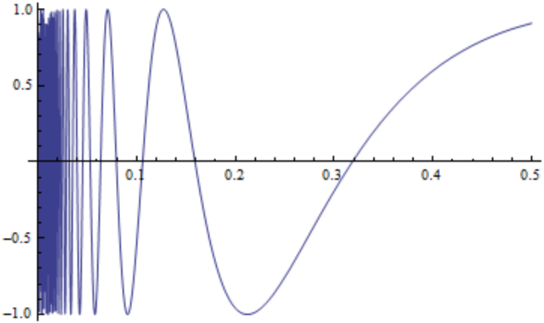
"Shape $\pi_i(C)$" vanishes.

If C is an ANR, this is equivalent to

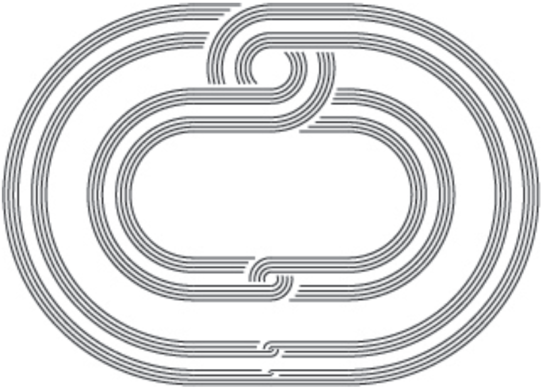
$\pi_i(C) = 0$ for $0 \leq i \leq k$.

Examples:
Topologists' sine curve and the Whitehead continuum are UV^k for all $k \geq 0$.

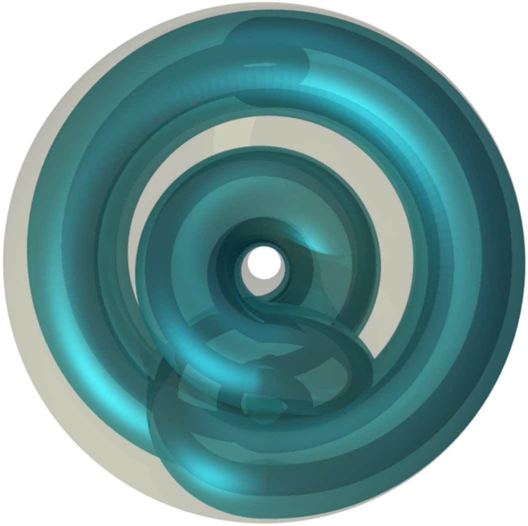
That is, they are **cell-like**.



Topologists' Sine Curve



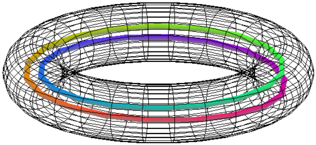
Whitehead Continuum - 4 stages of construction



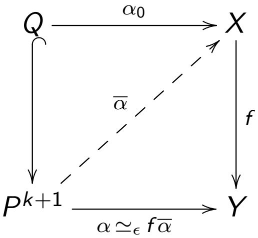
The dyadic solenoid

$$\Sigma = \operatorname{proj}\lim\{S^1 \to S^1 : z \mapsto z^2\}$$

$\pi_1(\Sigma) = 0$ and $\check{\pi}_1(\Sigma) = 0$, but
Σ is not UV^1.



A map $f\colon X \to Y$ between compact ANR's is UV^k if each point-inverse is UV^k.
Condition equivalent to having the ϵ-lifting property for polyhedra of dimension $\leq k+1$ for every $\epsilon > 0$:



A UV^k-map between compact ANR's induces an isomorphism on π_i, $0 \leq i \leq k$, and an epimorphism on π_{k+1}.

Around 1986, Bestvina and Walsh proved:

Theorem. Suppose M^n is a compact manifold and K is a polyhedron. If $f\colon M \to K$ is a $(k+1)$-connected map, then f is homotopic to a UV^k-map, provided that $2k+3 \leq n$.

Example: Any map f from the n-sphere to the m-sphere, where $5 \leq n \leq m$, is homotopic to a UV^1-map.

The inequality is best possible:

e.g., $f\colon S^n \to S^n$ may have degree $\neq \pm 1$.

Any improvement would imply f is a homotopy equivalence.

Other results of this type are due to Anderson (1956), Wilson (1970, 1973), Walsh (1975), Ferry (1994).

Černavskii proved a "controlled"

Theorem. If M is an n-manifold and $k \leq \left[\frac{n-3}{2}\right]$, then there is a UV^k map $p\colon M \to M \times I$. Moreover, for any $\epsilon > 0$, we can choose p so that $proj \circ p$ is ϵ-close to $1_M\colon M \to M$.

The control is with respect to the projection map $M \times I \to M$.

A map $f\colon X \to B$ between compact ANR's has property $UV^k(\epsilon)$ if it satisfies the ϵ-lifting property for $(k+1)$-dimensional polyhedra for a fixed $\epsilon > 0$.
If $p\colon B \to Y$ is a map to a metric space, then f has property $UV^k(\epsilon)$ **over B** if ϵ-liftings exist as measured in Y.
A compact ANR X has the **(linear) $\mathbf{UV^k}$-approximation property** if for every $\epsilon > 0$ there is a $\delta(= c \cdot \epsilon) > 0$ (c is a constant depending only on k) such that if
B compact, metric ANR,
$p\colon B \to Y$ is a map to a metric space, and
$f\colon X \to B$, is $UV^k(\delta)$ over Y,
then f is ϵ-homotopic over Y to a UV^k-map.
Theorem (Lacher). A map $f\colon A \to B$ between compact ENR's is UV^k iff it is $UV^k(\epsilon)$ for every $\epsilon > 0$.

Main Theorem. If X is a compact, connected ENR with the disjoint $(k+1)$-disks property, then X has the linear UV^k-approximation property.
Primary example: X is an ENR homology n-manifold, $n \geq 5$, with the disjoint disks property.

X has the disjoint $\frac{n-1}{2}$-disks property.

Corollary. An ENR homology n-manifold, $n \geq 5$, with the DDP has the $UV^{\frac{n-3}{2}}$-approximation property.

A **homology n-manifold** is a space X having the property that for each $x \in X$,

$$H_k(X, X - x; \mathbb{Z}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

A **euclidean neighborhood retract** (ENR) is a space homeomorphic to a closed subset of euclidean space that is a retract of some neighborhood of itself, that is, a locally compact, finite dimensional ANR.

A space X satisfies the **disjoint (k-)disks property**, or *DDP* (DDP^k), if any two maps of the 2-cell (k-cell) into X can be approximated by maps with disjoint images.

Examples:

1. Let $p\colon S^n \to S^{n+1}$, $n \geq 5$, be a Černavskii map, and let $f\colon S^n \to S^n$ be a μ-equivalence over S^{n+1} representing a non-zero element of

$$\mathcal{S}_\epsilon\begin{pmatrix} S^n \\ \downarrow p \\ S^{n+1} \end{pmatrix} \cong \mathbb{Z}$$

Then f is $(C \cdot \mu)$-homotopic **over $\mathbf{S^{n+1}}$** to a $UV^{\frac{n-3}{2}}$-map.
However, f is not controlled homotopic to a $UV^{\frac{n-1}{2}}$-map.
Otherwise, the resulting map would be cell-like (Lacher), hence, arbitrarily close to a homeomorphism.

2. Suppose $f\colon X \to Y$ is a homotopy equivalence between compact homology n-manifolds with the DDP, $n \geq 5$. Then f is homotopic to a $UV^{\frac{n-3}{2}}$-map.

If the Quinn indices

$$\iota(X) \neq \iota(Y) \in 1 + 8\mathbb{Z},$$

then f is not homotopic to a $UV^{\frac{n-1}{2}}$-map.

Otherwise, the resulting map would again be cell-like.

Given Y there is an $\epsilon > 0$ such that if $f\colon X \to Y$ is an ϵ-equivalence over Y, then $\iota(X) = \iota(Y)$.

What then?

CE-Approximation Conjecture. *If Y is a compact ENR homology n-manifold, $n \geq 5$, and $\epsilon > 0$ then there exists $\delta > 0$ such that if X is a compact ENR homology n-manifold with the DDP and $f : X \to Y$ is a δ-homotopy equivalence, then f is ϵ-homotopic to a cell-like map.*

The CE-approximation conjecture can be used to establish a version of the Chapman-Ferry α-approximation theorem for homology manifolds.

α-Approximation Conjecture for Homology Manifolds. *Given a compact ENR homology n-manifold Y with the DDP and $\epsilon > 0$ there is a $\delta > 0$ such that if X is a compact ENR homology n-manifold with the DDP and $f : X \to Y$ is a δ-equivalence, the f is ϵ-homotopic to a homeomorphism.*

It is not difficult to show that the α-approximation conjecture implies that an ENR homology n-manifold, $n \geq 5$, with the disjoint disks property is topologically homogeneous.

Given a $UV^k(\epsilon)$-map
$f\colon X \to B$
I. Find a "simple homotopy solution" to the problem,

$$X \nearrow \bar{X} \xrightarrow{f_1} B\ ,$$

where f_1 is $UV^k(\mu)$.

II. Get an $UV^k(\eta)$-homotopy inverse $g\colon X \to \bar{X}$ to the collapse $\bar{X} \searrow X$, where η is small enough so that the composition

$$X \xrightarrow{g} \bar{X} \xrightarrow{f_1} B$$

is (almost) $UV^k(\mu)$.

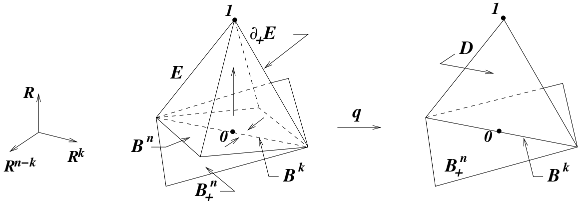
If a $(k+1)$-cell D is attached to $\mathbb{R}^n$ along B^k, $2k+3 \leq n$, then there is a homotopy of the inclusion

$\mathbb{R}^n \subseteq \mathbb{R}^n \cup_{B^k} D$

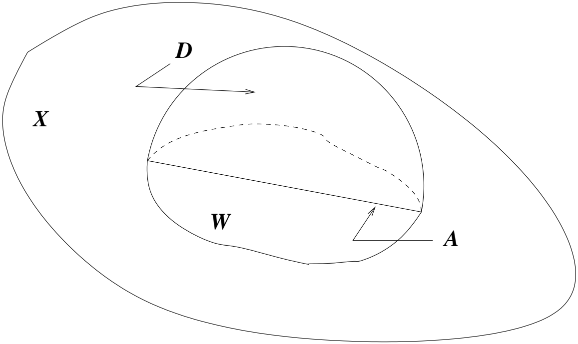
to a UV^k-map

$q\colon \mathbb{R}^n \to \mathbb{R}^n \cup_{B^k} D$

that is fixed outside a relative regular nbd B^n_+ of B^k.



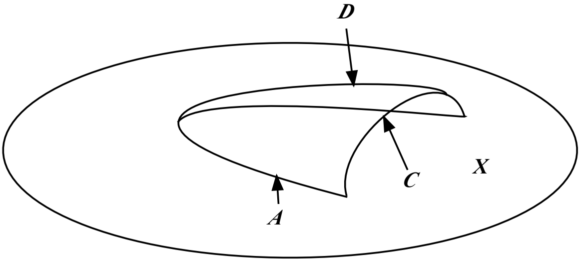
Main Lemma. A $UV^k(\delta)$ version of this holds for ENR's with the DDP^{k+1}, for every $\delta > 0$.



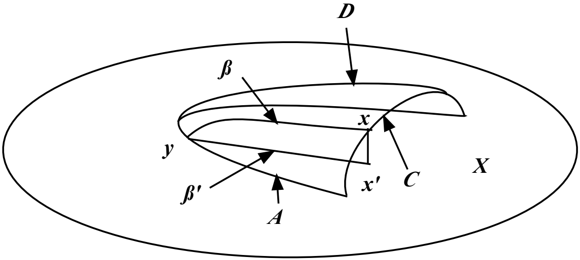
Key Fact: If X is an ENR with the DDP^{k+1}, then every map $\alpha \colon P \to X$ of a $(k+1)$-dimensional polyhedron can be approximated by an LCC^k embedding (rel any subpolyhedron on which α is already an LCC^k embedding).

I. A simple homotopy solution for $k = 0$.
Given $f\colon X \to B$ $UV^0(\epsilon)$.

Attach a finite number of arcs to X to get X_1 and an extension of f to X_1 that is $UV^0(\mu)$.
We've changed the homotopy type of X. Attach 2-cells to recover.



We've now lost $UV^0(\mu)$! But that can also be recovered.



The path β' comes from a μ-lift of β to X_1 and may wind around other "C" curves, so the picture is somewhat misleading.
II. Slide X onto $\bar{X}$ with arbitrary preassigned UV^0 control.