On Lusternik-Schnirelmann category

A. Dranishnikov

Department of Mathematics
University of Florida

FSU, February 22, 2013

Lusternik-Schnirelmann category

- $\mathrm{cat}_{\mathrm{LS}} X \leq n$ if there is an open covering $U_0, \dots, U_n$ by contractible in X sets (X-contractible).
- $\mathrm{cat}_{\mathrm{LS}}$ is a homotopy invariant.

- **Lusternik-Schnirelmann THEOREM:**

  $\mathrm{cat}_{\mathrm{LS}} M + 1 \leq Crit(f)$ *for any smooth function on a manifold M.*

- REMARK. For a Morse function the low bound is the sum of the Betti numbers.

Lusternik-Schnirelmann category

- $\mathrm{cat}_{\mathrm{LS}} X \leq n$ if there is an open covering $U_0, \dots, U_n$ by contractible in X sets (X-contractible).
- $\mathrm{cat}_{\mathrm{LS}}$ is a homotopy invariant.

- Lusternik-Schnirelmann THEOREM:

  $\mathrm{cat}_{\mathrm{LS}} M + 1 \leq Crit(f)$ *for any smooth function on a manifold M.*

  - REMARK. For a Morse function the low bound is the sum of the Betti numbers.

Lusternik-Schnirelmann category

- $\operatorname{cat}_{LS} X \leq n$ if there is an open covering $U_0, \ldots, U_n$ by contractible in X sets (X-contractible).
- cat_{LS} is a homotopy invariant.

Lusternik-Schnirelmann THEOREM:

$\operatorname{cat}_{LS} M + 1 \leq Crit(f)$ for any smooth function on a manifold M.

- REMARK. For a Morse function the low bound is the sum of the Betti numbers.

Lusternik-Schnirelmann category

- $\mathrm{cat}_{\mathrm{LS}} X \leq n$ if there is an open covering $U_0, \dots, U_n$ by contractible in X sets (X-contractible).
- $\mathrm{cat}_{\mathrm{LS}}$ is a homotopy invariant.

Lusternik-Schnirelmann THEOREM:

$\mathrm{cat}_{\mathrm{LS}} M + 1 \leq Crit(f)$ for any smooth function on a manifold M.

- REMARK. For a Morse function the low bound is the sum of the Betti numbers.

Applications

Lusternik-Schnirelmann theorem (1929).

Solution to Poincare Problem (1905): *Every Riemannian metric on S^2 has at least three closed geodesics.*

The minimum (=3) occurs on ellipsoids.

Birkhoff (1927). *There is at least one closed geodesic.*

Applications

Lusternik-Schnirelmann theorem (1929).

Solution to Poincare Problem (1905): *Every Riemannian metric on S^2 has at least three closed geodesics.*

The minimum (=3) occurs on ellipsoids.

Birkhoff (1927). *There is at least one closed geodesic.*

Applications

Lusternik-Schnirelmann theorem (1929).

Solution to Poincare Problem (1905): *Every Riemannian metric on S^2 has at least three closed geodesics.*

The minimum (=3) occurs on ellipsoids.

Birkhoff (1927). *There is at least one closed geodesic.*

Applications

Lusternik-Schnirelmann theorem (1929).

Solution to Poincare Problem (1905): *Every Riemannian metric on S^2 has at least three closed geodesics.*

The minimum (=3) occurs on ellipsoids.

Birkhoff (1927). *There is at least one closed geodesic.*

Applications

Lusternik-Schnirelmann theorem (1929).

Solution to Poincare Problem (1905): *Every Riemannian metric on S^2 has at least three closed geodesics.*

The minimum (=3) occurs on ellipsoids.

Birkhoff (1927). *There is at least one closed geodesic.*

Applications

Theorem

(Lusternik-Schnirelmann, 1929) *For every cover of S^n by $n+1$ open sets one of the sets contains an antipodal pair of points.*

Follows from the computation $\mathrm{cat}_{\mathrm{LS}}\,\mathbb{R}P^n = n$: If $U_0, \dots, U_n$ a cover of S^n without antipodal pairs, then $q(U_1), \dots, q_n(U_n)$ is a cover of $\mathbb{R}P^n$ where $q : S^n \to \mathbb{R}P^n$ is the quotient map. Moreover, $q|_{U_i} : U_i \to q(U_i)$ is a homeomorphism. Since U_i are contractible in S^n, $q(U_i)$ are contractible in $\mathbb{R}P^n$. Then $\mathrm{cat}_{\mathrm{LS}}\,\mathbb{R}P^n \le n-1$. Contradiction.

The Theorem is a reformulation of the Borsuk-Ulam theorem rediscovered in mid 30s: *For every continuous map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.*

Applications

Theorem

(Lusternik-Schnirelmann, 1929) *For every cover of S^n by $n+1$ open sets one of the sets contains an antipodal pair of points.*

Follows from the computation $\operatorname{cat}_{\mathrm{LS}} \mathbb{R}P^n = n$: If $U_0, \dots, U_n$ a cover of S^n without antipodal pairs, then $q(U_1), \dots, q_n(U_n)$ is a cover of $\mathbb{R}P^n$ where $q : S^n \to \mathbb{R}P^n$ is the quotient map. Moreover, $q|_{U_i} : U_i \to q(U_i)$ is a homeomorphism. Since U_i are contractible in S^n, $q(U_i)$ are contractible in $\mathbb{R}P^n$. Then $\operatorname{cat}_{\mathrm{LS}} \mathbb{R}P^n \leq n-1$. Contradiction.

The Theorem is a reformulation of the Borsuk-Ulam theorem rediscovered in mid 30s: *For every continuous map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.*

Applications

Theorem

(Lusternik-Schnirelmann, 1929) *For every cover of S^n by $n+1$ open sets one of the sets contains an antipodal pair of points.*

Follows from the computation $\operatorname{cat}_{\mathrm{LS}} \mathbb{R}P^n = n$: If $U_0, \dots, U_n$ a cover of S^n without antipodal pairs, then $q(U_1), \dots, q_n(U_n)$ is a cover of $\mathbb{R}P^n$ where $q : S^n \to \mathbb{R}P^n$ is the quotient map. Moreover, $q|_{U_i} : U_i \to q(U_i)$ is a homeomorphism. Since U_i are contractible in S^n, $q(U_i)$ are contractible in $\mathbb{R}P^n$. Then $\operatorname{cat}_{\mathrm{LS}} \mathbb{R}P^n \leq n-1$. Contradiction.

The Theorem is a reformulation of the Borsuk-Ulam theorem rediscovered in mid 30s: *For every continuous map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.*

Applications

Theorem

(Lusternik-Schnirelmann, 1929) *For every cover of S^n by $n+1$ open sets one of the sets contains an antipodal pair of points.*

Follows from the computation $\mathrm{cat}_{\mathrm{LS}}\, \mathbb{R}P^n = n$: If $U_0, \dots, U_n$ a cover of S^n without antipodal pairs, then $q(U_1), \dots, q_n(U_n)$ is a cover of $\mathbb{R}P^n$ where $q : S^n \to \mathbb{R}P^n$ is the quotient map. Moreover, $q|_{U_i} : U_i \to q(U_i)$ is a homeomorphism. Since U_i are contractible in S^n, $q(U_i)$ are contractible in $\mathbb{R}P^n$. Then $\mathrm{cat}_{\mathrm{LS}}\, \mathbb{R}P^n \leq n-1$. Contradiction.

The Theorem is a reformulation of the Borsuk-Ulam theorem rediscovered in mid 30s: *For every continuous map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ with $f(x) = f(-x)$.*

Some examples

- $\mathrm{cat}_{\mathrm{LS}} X = 0 \Leftrightarrow X$ is contractible.
- $\mathrm{cat}_{\mathrm{LS}} S^n = 1$.
- $\mathrm{cat}_{\mathrm{LS}}(S^1 \times S^1) = 2$.

Some examples

- $\mathrm{cat_{LS}}\, X = 0 \Leftrightarrow X$ is contractible.
- $\mathrm{cat_{LS}}\, S^n = 1$.
- $\mathrm{cat_{LS}}(S^1 \times S^1) = 2$.

Some examples

- $\mathrm{cat}_{\mathrm{LS}}\, X = 0 \Leftrightarrow X$ is contractible.
- $\mathrm{cat}_{\mathrm{LS}}\, S^n = 1$.
- $\mathrm{cat}_{\mathrm{LS}}(S^1 \times S^1) = 2$.

Some properties

- $\mathrm{cat}_{\mathrm{LS}} \leq$ dim.
- the cup-length $\leq \mathrm{cat}_{\mathrm{LS}}$.
- EXAMPLE (LS-theorem):

$$\mathrm{cat}_{\mathrm{LS}} \mathbb{R}P^n = n.$$

Some properties

- $\mathrm{cat}_{\mathrm{LS}} \leq$ dim.
- the cup-length $\leq \mathrm{cat}_{\mathrm{LS}}$.
- EXAMPLE (LS-theorem):

$$\mathrm{cat}_{\mathrm{LS}} \mathbb{R}P^n = n.$$

Some properties

- $\mathrm{cat}_{\mathrm{LS}} \leq$ dim.
- the cup-length $\leq \mathrm{cat}_{\mathrm{LS}}$.
- EXAMPLE (LS-theorem):

$$\mathrm{cat}_{\mathrm{LS}}\, \mathbb{R}P^n = n.$$

Manifolds with small category

Theorem

If $\mathrm{cat}_{\mathrm{LS}} M = 1$ *for a closed n-manifold, then* $M = S^n$*.*

Theorem (D.-Katz-Rudyak)

If $\mathrm{cat}_{\mathrm{LS}} M = 2$ *for a closed n-manifold,* $n \geq 3$*, then* $\pi_1(M)$ *is free.*

Manifolds with small category

Theorem

If $\mathrm{cat}_{\mathrm{LS}} M = 1$ *for a closed n-manifold, then* $M = S^n$*.*

Theorem (D.-Katz-Rudyak)

If $\mathrm{cat}_{\mathrm{LS}} M = 2$ *for a closed n-manifold,* $n \geq 3$*, then* $\pi_1(M)$ *is free.*

Manifolds with small category

Theorem

If $\operatorname{cat}_{\mathrm{LS}} M = 1$ *for a closed n-manifold, then* $M = S^n$.

Theorem (D.-Katz-Rudyak)

If $\operatorname{cat}_{\mathrm{LS}} M = 2$ *for a closed n-manifold,* $n \geq 3$*, then* $\pi_1(M)$ *is free.*

$\mathrm{cat}_{\mathrm{LS}} = 3$

Theorem (D.-Katz-Rudyak)

For every nonfree f.p. group π there is a 4-manifold M with $\pi_1(M) = \pi$ and $\mathrm{cat}_{\mathrm{LS}}\, M = 3$.

Definition of the BS-class

- The Berstein-Šchwarz class $\mathfrak{b}_\pi \in H^1(\pi; I(\pi))$ of a group π is the image of the generator under connecting homomorphism $H^0(\pi;\mathbb{Z}) \to H^1(\pi; I(\pi))$ in the long exact sequence generated by the short exact sequence of coefficients
$$0 \to I(\pi) \to \mathbb{Z}(\pi) \to \mathbb{Z} \to 0.$$
- For a complex N with a map $f : N \to B\pi$ classifying the universal covering, the BS-class of N is $\mathfrak{b}_N = f^*\mathfrak{b}_\pi$.

Definition of the BS-class

- The Berstein-Šchwarz class $\mathfrak{b}_\pi \in H^1(\pi; I(\pi))$ of a group π is the image of the generator under connecting homomorphism $H^0(\pi;\mathbb{Z}) \to H^1(\pi; I(\pi))$ in the long exact sequence generated by the short exact sequence of coefficients
$$0 \to I(\pi) \to \mathbb{Z}(\pi) \to \mathbb{Z} \to 0.$$
- For a complex N with a map $f : N \to B\pi$ classifying the universal covering, the BS-class of N is $\mathfrak{b}_N = f^*\mathfrak{b}_\pi$.

Universality of the BS-class

The cup product $\alpha \smile \beta \in H^{p+q}(X; A \otimes B)$ is defined for $\alpha \in H^p(X; A)$ and $\beta \in H^q(X; B)$ for any π-modules A and B where $\pi = \pi_1(X)$.

Universality Theorem

For every π-module L, every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\mathfrak{b}_\pi)^k$ under a suitable coefficients homomorphism $\psi : I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L$.

Corollary

$$cd(\pi) = \max\{n \mid \mathfrak{b}_\pi^n \neq 0\}.$$

Universality of the BS-class

The cup product $\alpha \smile \beta \in H^{p+q}(X; A \otimes B)$ is defined for $\alpha \in H^p(X; A)$ and $\beta \in H^q(X; B)$ for any π-modules A and B where $\pi = \pi_1(X)$.

Universality Theorem

For every π-module L, every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\mathfrak{b}_\pi)^k$ under a suitable coefficients homomorphism $\psi : I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L$.

Corollary

$$cd(\pi) = \max\{n \mid \mathfrak{b}_\pi^n \neq 0\}.$$

Universality of the BS-class

The cup product $\alpha \smile \beta \in H^{p+q}(X; A \otimes B)$ is defined for $\alpha \in H^p(X; A)$ and $\beta \in H^q(X; B)$ for any π-modules A and B where $\pi = \pi_1(X)$.

Universality Theorem

For every π-module L, every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\mathfrak{b}_\pi)^k$ under a suitable coefficients homomorphism $\psi : I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L$.

Corollary

$$cd(\pi) = \max\{n \mid \mathfrak{b}_\pi^n \neq 0\}.$$

Universality of the BS-class

The cup product $\alpha \smile \beta \in H^{p+q}(X; A \otimes B)$ is defined for $\alpha \in H^p(X; A)$ and $\beta \in H^q(X; B)$ for any π-modules A and B where $\pi = \pi_1(X)$.

Universality Theorem

For every π-module L, every cohomology class $\alpha \in H^k(\pi; L)$ is the image of $(\mathfrak{b}_\pi)^k$ under a suitable coefficients homomorphism $\psi : I(\pi)^k = I(\pi) \otimes \cdots \otimes I(\pi) \to L$.

Corollary

$$cd(\pi) = \max\{n \mid \mathfrak{b}_\pi^n \neq 0\}.$$

Proof of $\mathrm{cat}_{\mathrm{LS}} M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\mathrm{cat}_{\mathrm{LS}} M \geq 3$ by the cup-length inequality. Contradiction.

Proof of $\operatorname{cat}_{LS} M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\operatorname{cat}_{LS} M \geq 3$ by the cup-length inequality. Contradiction.

Proof of $\mathrm{cat}_{\mathrm{LS}} M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\mathrm{cat}_{\mathrm{LS}} M \geq 3$ by the cup-length inequality. Contradiction.

Proof of $\mathrm{cat}_{\mathrm{LS}} M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\mathrm{cat}_{\mathrm{LS}} M \geq 3$ by the cup-length inequality. Contradiction.

Proof of $\mathrm{cat}_{\mathrm{LS}} M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\mathrm{cat}_{\mathrm{LS}} M \geq 3$ by the cup-length inequality. Contradiction.

Proof of $\mathrm{cat}_{\mathrm{LS}}\, M = 2$ Theorem

- Assume that $\pi = \pi_1(M)$ is not free. Then $cd(\pi) \geq 2$.
- Consider $f : M \to B\pi$ that induces iso on π_1.
- Then $f^* : H^2(B\pi; L) \to H^2(M; L)$ is mono.
- Then $\mathfrak{b}_M^2 \neq 0$ for $\mathfrak{b}_M = f^*(\mathfrak{b}_\pi)$.
- By the (twisted) PD there is α such that $\mathfrak{b}_M^2 \smile \alpha \neq 0$.
- Then $\mathrm{cat}_{\mathrm{LS}}\, M \geq 3$ by the cup-length inequality. Contradiction.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}} M \leq \dim M/2$ *for simply connected* M*.*

First proved by Grossman in 1940s

Rudyak's Conjecture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ *is free, then* $\mathrm{cat}_{\mathrm{LS}} M \leq \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}} M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}} M \leq \dim M/2$ *for simply connected M.*

First proved by Grossman in 1940s

Rudyak's Conjecture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ is free, then $\mathrm{cat}_{\mathrm{LS}} M \leq \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}} M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}}\, M \leq \dim M/2$ *for simply connected* M.

First proved by Grossman in 1940s

Rudyak's Conjecture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}}\, M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ *is free, then* $\mathrm{cat}_{\mathrm{LS}}\, M \leq \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}}\, M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2$ *for simply connected M.*

First proved by Grossman in 1940s

Rudyak's Conjcture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ is free, then $\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}} M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2$ *for simply connected M.*

First proved by Grossman in 1940s

Rudyak's Conjcture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ is free, then $\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}} M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}}\, M \leq \dim M/2$ *for simply connected* M.

First proved by Grossman in 1940s

Rudyak's Conjcture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}}\, M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ *is free, then* $\mathrm{cat}_{\mathrm{LS}}\, M \leq \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}}\, M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2$ *for simply connected* M.

First proved by Grossman in 1940s

Rudyak's Conjcture

If $\pi_1(M)$ is free, then the upper bound for $\mathrm{cat}_{\mathrm{LS}} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ *is free, then* $\mathrm{cat}_{\mathrm{LS}} M \le \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\mathrm{cat}_{\mathrm{LS}} M = n + 1 = [\dim M/2 + 1]$.

Simply connected spaces

Whitehead's Theorem.

$\operatorname{cat}_{LS} M \leq \dim M/2$ *for simply connected M.*

First proved by Grossman in 1940s

Rudyak's Conjcture

If $\pi_1(M)$ is free, then the upper bound for $\operatorname{cat}_{LS} M$ is of order $\dim M/2$.

Theorem

If $\pi_1(M)$ *is free, then* $\operatorname{cat}_{LS} M \leq \dim M/2 + 1$.

Example: $M = \mathbb{C}P^n \times S^1$, then $\operatorname{cat}_{LS} M = n + 1 = [\dim M/2 + 1]$.

Main Result

- **THEOREM.**

  For every complex X,

  $$\operatorname{cat}_{\mathrm{LS}} X \le cd(\pi_1(X)) + \left\lceil \frac{hd(X) - 1}{2} \right\rceil .$$

  Here $\lceil z \rceil$ is the smallest integer n with $z \le n$.

  - $hd(X)$ is the homotopical dimension of X.

Main Result

THEOREM.

For every complex X,

$$\mathrm{cat}_{\mathrm{LS}} X \le cd(\pi_1(X)) + \left\lceil \frac{hd(X) - 1}{2} \right\rceil .$$

Here $\lceil z \rceil$ is the smallest integer n with $z \le n$.

- $hd(X)$ is the homotopical dimension of X.

Analogy with dimension

- DEFINITION. *For a space X, $\dim X \leq n$ iff X admits an arbitrary small locally finite open cover $\mathcal{V}$ with $Ord\mathcal{V} \leq n+1$.*
- PROPOSITION. *For a space X, $\mathrm{cat}_{\mathrm{LS}} X \leq n$ iff X admits an X-contractible locally finite open cover $\mathcal{V}$ with $Ord\mathcal{V} \leq n+1$.*

Analogy with dimension

- DEFINITION. *For a space X, $\dim X \leq n$ iff X admits an arbitrary small locally finite open cover $\mathcal{V}$ with $Ord\mathcal{V} \leq n+1$.*
- PROPOSITION. *For a space X, $\operatorname{cat}_{\mathrm{LS}} X \leq n$ iff X admits an X-contractible locally finite open cover $\mathcal{V}$ with $Ord\mathcal{V} \leq n+1$.*

Hurewicz Mapping Theorem

- Hurewicz Theorem for dimension. For $f : X \to Y$,

$$\dim X \leq \dim Y + \max\{\dim f^{-1}(y)\}.$$

- In particular, $\dim(X \times Y) \leq \dim X + \dim Y$.
- For the product the $\mathrm{cat}_{\mathrm{LS}}$-analog holds true: $\mathrm{cat}_{\mathrm{LS}}(X \times Y) \leq \mathrm{cat}_{\mathrm{LS}} X + \mathrm{cat}_{\mathrm{LS}} Y$.
- What about Hurewicz for $\mathrm{cat}_{\mathrm{LS}}$? Does the inequality

$$\mathrm{cat}_{\mathrm{LS}} X \leq \mathrm{cat}_{\mathrm{LS}} Y + \mathrm{cat}_{\mathrm{LS}} F$$

hold for locally trivial bundles $X \xrightarrow{F} Y$?

Hurewicz Mapping Theorem

- Hurewicz Theorem for dimension. For $f : X \to Y$,

$$\dim X \leq \dim Y + \max\{\dim f^{-1}(y)\}.$$

- In particular, $\dim(X \times Y) \leq \dim X + \dim Y$.
- For the product the $\mathrm{cat}_{\mathrm{LS}}$-analog holds true: $\mathrm{cat}_{\mathrm{LS}}(X \times Y) \leq \mathrm{cat}_{\mathrm{LS}} X + \mathrm{cat}_{\mathrm{LS}} Y$.
- What about Hurewicz for $\mathrm{cat}_{\mathrm{LS}}$? Does the inequality

$$\mathrm{cat}_{\mathrm{LS}} X \leq \mathrm{cat}_{\mathrm{LS}} Y + \mathrm{cat}_{\mathrm{LS}} F$$

hold for locally trivial bundles $X \xrightarrow{F} Y$?

Hurewicz Mapping Theorem

- Hurewicz Theorem for dimension. For $f : X \to Y$,

$$\dim X \leq \dim Y + \max\{\dim f^{-1}(y)\}.$$

- In particular, $\dim(X \times Y) \leq \dim X + \dim Y$.
- For the product the $\mathrm{cat_{LS}}$-analog holds true: $\mathrm{cat_{LS}}(X \times Y) \leq \mathrm{cat_{LS}}\, X + \mathrm{cat_{LS}}\, Y$.
- What about Hurewicz for $\mathrm{cat_{LS}}$? Does the inequality

$$\mathrm{cat_{LS}}\, X \leq \mathrm{cat_{LS}}\, Y + \mathrm{cat_{LS}}\, F$$

hold for locally trivial bundles $X \xrightarrow{F} Y$?

Hurewicz Mapping Theorem

- Hurewicz Theorem for dimension. For $f : X \to Y$,

$$\dim X \leq \dim Y + \max\{\dim f^{-1}(y)\}.$$

- In particular, $\dim(X \times Y) \leq \dim X + \dim Y$.
- For the product the $\mathrm{cat}_{\mathrm{LS}}$-analog holds true: $\mathrm{cat}_{\mathrm{LS}}(X \times Y) \leq \mathrm{cat}_{\mathrm{LS}} X + \mathrm{cat}_{\mathrm{LS}} Y$.
- What about Hurewicz for $\mathrm{cat}_{\mathrm{LS}}$? Does the inequality

$$\mathrm{cat}_{\mathrm{LS}} X \leq \mathrm{cat}_{\mathrm{LS}} Y + \mathrm{cat}_{\mathrm{LS}} F$$

hold for locally trivial bundles $X \xrightarrow{F} Y$?

Hurewicz Mapping Theorem

- The answer is 'No': $f : RP^5 \to \mathbb{C}P^2$ with fiber S^1. Then $5 = \mathrm{cat}_{\mathrm{LS}}\, \mathbb{R}P^5 > \mathrm{cat}_{\mathrm{LS}}\, CP^2 + \mathrm{cat}_{\mathrm{LS}}\, S^1 = 2 + 1$.

CONJECTURE

$$\mathrm{cat}_{\mathrm{LS}}\, X \leq \dim Y + \mathrm{cat}_{\mathrm{LS}}\, F$$

for locally trivial bundles $X \xrightarrow{F} Y$.

CONJECTURE implies THEOREM

Apply the formula from CONJECTURE to f in the Borel construction

$$M \longleftarrow \widetilde{M} \times_\pi E\pi \xrightarrow{f} B\pi.$$

where $\pi = \pi_1(M)$ and $\widetilde{M}$ is the universal cover of M.
Then

$$\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) \leq \dim B\pi + \mathrm{cat_{LS}}\, \widetilde{M}.$$

Note that $\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) = \mathrm{cat_{LS}}\, M$. Also $\mathrm{cat_{LS}}\, \widetilde{M} \leq \dim M/2$ by the Whitehead theorem. Thus, $\mathrm{cat_{LS}}\, M \leq cd(\pi) + \dim M/2$ if Eilenberg-Ganea holds true.

CONJECTURE implies THEOREM

Apply the formula from CONJECTURE to f in the Borel construction

$$M \longleftarrow \widetilde{M} \times_\pi E\pi \xrightarrow{f} B\pi.$$

where $\pi = \pi_1(M)$ and $\widetilde{M}$ is the universal cover of M.
Then

$$\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) \le \dim B\pi + \mathrm{cat_{LS}}\, \widetilde{M}.$$

Note that $\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) = \mathrm{cat_{LS}}\, M$. Also $\mathrm{cat_{LS}}\, \widetilde{M} \le \dim M/2$ by the Whitehead theorem. Thus, $\mathrm{cat_{LS}}\, M \le cd(\pi) + \dim M/2$ if Eilenberg-Ganea holds true.

CONJECTURE implies THEOREM

Apply the formula from CONJECTURE to f in the Borel construction

$$M \longleftarrow \widetilde{M} \times_\pi E\pi \xrightarrow{f} B\pi.$$

where $\pi = \pi_1(M)$ and $\widetilde{M}$ is the universal cover of M.
Then

$$\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) \leq \dim B\pi + \mathrm{cat_{LS}}\,\widetilde{M}.$$

Note that $\mathrm{cat_{LS}}(\widetilde{M} \times_\pi E\pi) = \mathrm{cat_{LS}}\,M$. Also $\mathrm{cat_{LS}}\,\widetilde{M} \leq \dim M/2$ by the Whitehead theorem. Thus, $\mathrm{cat_{LS}}\,M \leq cd(\pi) + \dim M/2$ if Eilenberg-Ganea holds true.

CONJECTURE implies THEOREM

Apply the formula from CONJECTURE to f in the Borel construction

$$M \longleftarrow \widetilde{M} \times_\pi E\pi \xrightarrow{f} B\pi.$$

where $\pi = \pi_1(M)$ and $\widetilde{M}$ is the universal cover of M.
Then

$$\mathrm{cat}_{\mathrm{LS}}(\widetilde{M} \times_\pi E\pi) \leq \dim B\pi + \mathrm{cat}_{\mathrm{LS}} \widetilde{M}.$$

Note that $\mathrm{cat}_{\mathrm{LS}}(\widetilde{M} \times_\pi E\pi) = \mathrm{cat}_{\mathrm{LS}} M$. Also $\mathrm{cat}_{\mathrm{LS}} \widetilde{M} \leq \dim M/2$ by the Whitehead theorem. Thus, $\mathrm{cat}_{\mathrm{LS}} M \leq cd(\pi) + \dim M/2$ if Eilenberg-Ganea holds true.

Hilbert's 13th problem

- Hilbert-13 (1900): "Prove that the equation of seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable by means of any continuous functions of only two variables." $\Leftrightarrow$ Each of the implicit functions $x = x(a, b, c)$ is not representable as a composition of functions of two variables.
- Kolmogorov's Superposition theorem (1957). *Every function of n variables can be represented as a composition of functions of two variables.*
- Ostrand (1960) Kolmogorov's result is of dimension-theoretical nature.

Hilbert's 13th problem

- Hilbert-13 (1900): "Prove that the equation of seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable by means of any continuous functions of only two variables."
  $\Leftrightarrow$ Each of the implicit functions $x = x(a, b, c)$ is not representable as a composition of functions of two variables.
- Kolmogorov's Superposition theorem (1957). *Every function of n variables can be represented as a composition of functions of two variables.*
- Ostrand (1960) Kolmogorov's result is of dimension-theoretical nature.

Hilbert's 13th problem

- Hilbert-13 (1900): "Prove that the equation of seventh degree $x^7 + ax^3 + bx^2 + cx + 1 = 0$ is not solvable by means of any continuous functions of only two variables." $\Leftrightarrow$ Each of the implicit functions $x = x(a, b, c)$ is not representable as a composition of functions of two variables.
- Kolmogorov's Superposition theorem (1957). *Every function of n variables can be represented as a composition of functions of two variables.*
- Ostrand (1960) Kolmogorov's result is of dimension-theoretical nature.

Analogy with dimension

Ostrand: A cover $\mathcal{U} = \{U_i\}$ is called an *n*-cover if every *n* elements of $\mathcal{U}$ form a cover.

Kolmogorov-Ostrand's Theorem. $\dim X \leq n \Leftrightarrow$ *for any open cover* $\mathcal{V}$ *and for every* $m \geq n$ *there is an* $(n+1)$*-cover* $U_0, \dots, U_m$ *such that* $U_i = \coprod U_i^\alpha$ *where* $\{U_i^\alpha\}_{i,\alpha} \prec \mathcal{V}$.

$\mathrm{cat}_{\mathrm{LS}}$-Analog of Kolmogorov-Ostrand Theorem.

$\mathrm{cat}_{\mathrm{LS}} X \leq n \Leftrightarrow$ *for any* $m \geq n$ *there is an* $(n+1)$*-cover* $U_0, \dots, U_m$ *by open* X*-contractible sets.*

Analogy with dimension

Ostrand: A cover $\mathcal{U} = \{U_i\}$ is called an *n*-cover if every *n* elements of $\mathcal{U}$ form a cover.

Kolmogorov-Ostrand's Theorem. $\dim X \leq n \Leftrightarrow$ *for any open cover* $\mathcal{V}$ *and for every* $m \geq n$ *there is an* $(n+1)$*-cover* $U_0, \dots, U_m$ *such that* $U_i = \coprod U_i^\alpha$ *where* $\{U_i^\alpha\}_{i,\alpha} \prec \mathcal{V}$.

cat_{LS}-Analog of Kolmogorov-Ostrand Theorem.

$\mathrm{cat}_{LS} X \leq n \Leftrightarrow$ *for any* $m \geq n$ *there is an* $(n+1)$*-cover* $U_0, \dots, U_m$ *by open* X*-contractible sets.*

Analogy with dimension

Ostrand: A cover $\mathcal{U} = \{U_i\}$ is called an n-cover if every n elements of $\mathcal{U}$ form a cover.

Kolmogorov-Ostrand's Theorem. *$\dim X \le n \Leftrightarrow$ for any open cover $\mathcal{V}$ and for every $m \ge n$ there is an $(n+1)$-cover $U_0, \dots, U_m$ such that $U_i = \coprod U_i^\alpha$ where $\{U_i^\alpha\}_{i,\alpha} \prec \mathcal{V}$.*

$\mathrm{cat}_{\mathrm{LS}}$-Analog of Kolmogorov-Ostrand Theorem.

$\mathrm{cat}_{\mathrm{LS}} X \le n \Leftrightarrow$ for any $m \ge n$ there is an $(n+1)$-cover $U_0, \dots, U_m$ by open X-contractible sets.

Kolmogorov's trick

Proposition

A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$-cover of X if and only if $Ord_x\mathcal{U} \geq m - n$ for all $x \in X$.

Here $Ord_x\mathcal{U}$ is the number of elements of $\mathcal{U}$ that contain x

Proof of $\mathrm{cat_{LS}}\, X \times Y \leq \mathrm{cat_{LS}}\, X + \mathrm{cat_{LS}}\, Y$:

Let $\mathrm{cat_{LS}}\, X = n$ and $\mathrm{cat_{LS}}\, Y = m$. Let $\mathcal{U} = \{U_0, \dots, U_{m+n+1}\}$ be an $(n+1)$-cover of X and let $\mathcal{V} = \{V_0, \dots, V_{m+n+1}\}$ be an $(m+1)$-cover of Y by X and Y-contractible sets.

Claim: $\mathcal{W} = \{U_i \times V_i\}_{i=0}^{m+n+1}$ is a cover of $X \times Y$. Indeed, given (x, y), point x is covered by $m + n + 1 - n$ sets. Then the corresponding $U_i \times V_i$ cover $x \times Y$.

Kolmogorov's trick

Proposition

A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$-cover of X if and only if $Ord_x\mathcal{U} \geq m - n$ for all $x \in X$.

Here $Ord_x\mathcal{U}$ is the number of elements of $\mathcal{U}$ that contain x

Proof of $\mathrm{cat_{LS}}\, X \times Y \leq \mathrm{cat_{LS}}\, X + \mathrm{cat_{LS}}\, Y$:

Let $\mathrm{cat_{LS}}\, X = n$ and $\mathrm{cat_{LS}}\, Y = m$. Let $\mathcal{U} = \{U_0, \dots, U_{m+n+1}\}$ be an $(n+1)$-cover of X and let $\mathcal{V} = \{V_0, \dots, V_{m+n+1}\}$ be an $(m+1)$-cover of Y by X and Y-contractible sets.

Claim: $\mathcal{W} = \{U_i \times V_i\}_{i=0}^{m+n+1}$ is a cover of $X \times Y$. Indeed, given (x, y), point x is covered by $m + n + 1 - n$ sets. Then the corresponding $U_i \times V_i$ cover $x \times Y$.

Kolmogorov's trick

Proposition

A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$-cover of X if and only if $Ord_x\mathcal{U} \geq m - n$ for all $x \in X$.

Here $Ord_x\mathcal{U}$ is the number of elements of $\mathcal{U}$ that contain x

Proof of $\operatorname{cat}_{\mathrm{LS}} X \times Y \leq \operatorname{cat}_{\mathrm{LS}} X + \operatorname{cat}_{\mathrm{LS}} Y$:

Let $\operatorname{cat}_{\mathrm{LS}} X = n$ and $\operatorname{cat}_{\mathrm{LS}} Y = m$. Let $\mathcal{U} = \{U_0, \dots, U_{m+n+1}\}$ be an $(n+1)$-cover of X and let $\mathcal{V} = \{V_0, \dots, V_{m+n+1}\}$ be an $(m+1)$-cover of Y by X and Y-contractible sets.

Claim: $\mathcal{W} = \{U_i \times V_i\}_{i=0}^{m+n+1}$ is a cover of $X \times Y$. Indeed, given (x, y), point x is covered by $m + n + 1 - n$ sets. Then the corresponding $U_i \times V_i$ cover $x \times Y$.

Kolmogorov's trick

Proposition

A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$-cover of X if and only if $Ord_x\mathcal{U} \geq m - n$ for all $x \in X$.

Here $Ord_x\mathcal{U}$ is the number of elements of $\mathcal{U}$ that contain x

Proof of $\operatorname{cat}_{\mathrm{LS}} X \times Y \leq \operatorname{cat}_{\mathrm{LS}} X + \operatorname{cat}_{\mathrm{LS}} Y$:

Let $\operatorname{cat}_{\mathrm{LS}} X = n$ and $\operatorname{cat}_{\mathrm{LS}} Y = m$. Let $\mathcal{U} = \{U_0, \dots, U_{m+n+1}\}$ be an $(n+1)$-cover of X and let $\mathcal{V} = \{V_0, \dots, V_{m+n+1}\}$ be an $(m+1)$-cover of Y by X and Y-contractible sets.

Claim: $\mathcal{W} = \{U_i \times V_i\}_{i=0}^{m+n+1}$ is a cover of $X \times Y$. Indeed, given (x, y), point x is covered by $m + n + 1 - n$ sets. Then the corresponding $U_i \times V_i$ cover $x \times Y$.

Ganea's Approach to $\mathrm{cat}_{\mathrm{LS}}$

- RECALL: Path fibration: $p : PX \to X$, PX is the path space on X (with fixed $x_0 \in X$), $p(\phi) = \phi(0)$. The fiber of p is the loop space ΩX.
  X is contractible $\Leftrightarrow$ there is a section $s : X \to PX$.
- The n-th Ganea fibration $p_n : G_n(X) \to X$ is the fiber-wise join of $n+1$ copies of the path fibration. Thus, the fiber $p_n^{-1}(x_0) = *^{n+1}\Omega X$.
- REMARK. $G_0 = PX$, $p_0 = p$, and the fiber is ΩX.

Ganea's Approach to $\mathrm{cat}_{\mathrm{LS}}$

- RECALL: Path fibration: $p : PX \to X$, PX is the path space on X (with fixed $x_0 \in X$), $p(\phi) = \phi(0)$. The fiber of p is the loop space ΩX.
  X is contractible $\Leftrightarrow$ there is a section $s : X \to PX$.
- The n-th Ganea fibration $p_n : G_n(X) \to X$ is the fiber-wise join of $n+1$ copies of the path fibration. Thus, the fiber $p_n^{-1}(x_0) = *^{n+1}\Omega X$.
- REMARK. $G_0 = PX$, $p_0 = p$, and the fiber is ΩX.

Ganea's Approach to $\mathrm{cat}_{\mathrm{LS}}$

- RECALL: Path fibration: $p : PX \to X$, PX is the path space on X (with fixed $x_0 \in X$), $p(\phi) = \phi(0)$. The fiber of p is the loop space ΩX.
  X is contractible $\Leftrightarrow$ there is a section $s : X \to PX$.
- The n-th Ganea fibration $p_n : G_n(X) \to X$ is the fiber-wise join of $n+1$ copies of the path fibration. Thus, the fiber $p_n^{-1}(x_0) = *^{n+1}\Omega X$.
- REMARK. $G_0 = PX$, $p_0 = p$, and the fiber is ΩX.

Ganea-Schwarz Approach to $\mathrm{cat}_{\mathrm{LS}}$

THEOREM [Ganea-Schwarz]

$\mathrm{cat}_{\mathrm{LS}}\, X \le n \Leftrightarrow p_n : G_n(X) \to X$ *admits a section.*

EXAMPLE ($n = 0$). $p_0 : G_0(X) = PX \to X$ admits a section if and only if X is contractible.

Ganea-Schwarz Approach to $\mathrm{cat}_{\mathrm{LS}}$

THEOREM [Ganea-Schwarz]

$\mathrm{cat}_{\mathrm{LS}} X \leq n \Leftrightarrow p_n : G_n(X) \to X$ *admits a section.*

EXAMPLE ($n = 0$). $p_0 : G_0(X) = PX \to X$ admits a section if and only if X is contractible.

Ganea-Schwarz Approach to $\mathrm{cat}_{\mathrm{LS}}$

THEOREM [Ganea-Schwarz]

$\mathrm{cat}_{\mathrm{LS}} X \leq n \Leftrightarrow p_n : G_n(X) \to X$ *admits a section.*

EXAMPLE ($n = 0$). $p_0 : G_0(X) = PX \to X$ admits a section if and only if X is contractible.

THANK YOU!!!