Essential manifolds and macroscopic dimension

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Essential Manifolds

- An n-manifold M is called *essential* if the classifying map $f : M \to B\pi$ of its universal covering $\widetilde{M}$ cannot be deformed to the $(n-1)$-skeleton.
- Otherwise it is called *inessential*.
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- Manifolds with positive scalar curvature (PSC) tend to be inessential.
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Go Macroscopic

- Gromov suggested that a proper formulation of his observation about PSC manifolds M should be in terms of universal covering $\widetilde{M}$:

$$PSC \Rightarrow dim_{mc}\widetilde{M} < dimM$$

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$$dim_{mc}X \leq k$$

  iff there is a uniformly cobounded map $\phi : X \to N^k$ to a k-dimensional simplicial complex.
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Properties of macroscopic dimension

- $dim_{mc}X \leq dimX$
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- $dim_{mc}\mathbb{R}^n = n$ and
- generally, $dim_{mc}V^n = n$ for every uniformly contractible manifold with proper metric.

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md-small vs inessential

- If M^n is inessential, then M^n is md-small.
- The universal covering map $p : \widetilde{M^n} \to M^n$ admits an extension to the Stone-Čech compactifcation $\bar{p} : \beta(\widetilde{M^n}) \to M^n$.
- PROPOSITION. M^n is md-small $\Leftrightarrow f \circ \bar{p} : \beta(\widetilde{M^n}) \to B\pi$ can be deformed to the $(n-1)$-skeleton.
- Here $f : M^n \to B\pi$ classifies the universal covering.

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Homological formulation of essentiality

Theorem

TFAE

- An orientable n-manifold M is inessential
- $f_*([M]) = 0$.

Rational essentiality

- DEFINITION. An orientable n-manifold M is *rationally inessential* if $f_*([M]) = 0$ in $H_*(B\pi;\mathbb{Q})$.

GROMOV's CONJECTURE:

Every md-small manifold is rationally inessential.

- The conjecture is from Gelfand-80 book [1996]

Modification of dim_{mc}

- $dim_{MC}X \leq k$ if there is a uniformly cobounded Lipschitz map $f : X \to K$ to a k-dimensional simplicial complex.
- $dim_{mc}X \leq dim_{MC}X \leq asdimX$ for all X.

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Main Result

THEOREM .

For all $n \geq 4$ there are rationally essential md-small n-manifolds.

THEOREM gives a counterexample to Gromov's Conjecture.

Obstruction to essentiality

Theorem.

Let $L = \pi_{n-1}(B\pi^{n-1})$ denote the corresponding π-module where $\pi = \pi_1(M^n)$. Let $\kappa \in H^n(B\pi; L)$ be the primary obstruction to the retraction of $B\pi$ to $B\pi^{n-1}$. TFAE

- M^n is inessential;
- $f^*(\kappa) = 0$;
- $f^*(\beta^n) = 0$ where $\beta \in H^1(\pi, I(\pi))$ is the Berstein-Schwartz class.
- $f_*([M]) = 0$.

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Equivariant vs almost equivariant

- Cohomology with local coefficients $H^n(X; L)$ are defined by means of equivariant cochains on $\widetilde{X}$. A homomorphism $\psi : C_n(\widetilde{X}) \to L$ is *equivariant* if the set

$$\{\gamma^{-1}\psi(\gamma e) \mid \gamma \in \pi\} \subset L$$

consists of one element $\psi(e)$ for every n-cell e in $\widetilde{X}$.

- A homomorphism $\psi : C_n(\widetilde{X}) \to L$ is called *almost equivariant*, if the set

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Almost equivariant cohomology

- Let $Hom_{ae}(C_n(\widetilde{X}), L)$ be the group of all almost equivariant homomorphisms from $C_n(\widetilde{X})$ to L.
- The homology groups of corresponding cochain complex $H^*_{ae}(\widetilde{X}; L)$ are called the *almost equivariant cohomology* of $\widetilde{X}$ with coefficients in a π-module L.

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Perturbation homomorphism

Since every equivariant homomorphism is almost equivariant, there is a natural transformation

$$pert_X^* : H^*(X; L) = H_\pi^*(\widetilde{X}; L) \to H_{ae}^*(\widetilde{X}; L)$$

called a *perturbation homomorphism* from the cohomology of X to the almost equivariant cohomology.

Primary obstruction to being md-small

- Let $f : M \to B\pi$ be a classifying $\widetilde{M}$ map and let $o_f \in H^n(M; \pi_{n-1}(B\pi^{(n-1)}))$ denote the primary obstruction to a deformation of M to the $(n-1)$-skeleton.
- Thus, o_f is the primary obstruction to essentiality.

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Let M be an n-manifold. Then the primary obstruction to $dim_{MC}\widetilde{M} < n$ is $pert_M^*(o_f) \in H^n_{ae}(\widetilde{M}; \pi_{n-1}(B\pi^{(n-1)}))$.

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Almost equivariant homology

- Let X be a CW complex with the universal cover $\widetilde{X}$ and let L be a π-module. We call an infinite chain $\sum_{e\in E_n(\widetilde{X})} \lambda_e e$ *almost equivariant* if the set $\{\gamma^{-1}\lambda_{\gamma e} \mid \gamma \in \pi\} \subset L$ is finite for every cell e. We call the homology defined by the almost equivariant locally finite chain *the almost equivariant locally finite homology* $H_*^{lf,ae}(\widetilde{X}; L)$.
- Note that the complex of equivariant locally finite chains defines equivariant locally finite homology $H_*^{lf,\pi}(\widetilde{X}; L)$. Then as in the case of cohomology for any complex K there is a *perturbation homomorphism*

$$pert_*^K : H_*(K; L) = H_*^{lf,\pi}(\widetilde{K}; L) \to H_*^{lf,ae}(\widetilde{K}; L).$$

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md-small homology

We define the group of *small macroscopic dimension classes* as $H_n^{sm}(\pi) = ker(pert_*^{\pi}) \subset H_n(\pi)$.

THEOREM.

For a closed oriented n-manifold M the following are equivalent:
1. M is md-small (i.e. $dim_{MC}\widetilde{M} < n$);
2. $f_*([M]) \in H_n^{sm}(\pi)$ where $f : M \to B\pi$ is the map classifying the universal covering $\widetilde{M}$ of M.

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Uniformly finite homology

When $L = \mathbb{Z}$ (or $\mathbb{R}$) is a trivial π-module, the almost equivariant locally finite homology groups $H_*^{lf,ae}(\widetilde{K}; L)$ coincide with the *uniformly finite homology* $H_*^{uf}(\widetilde{K}; \mathbb{Z})$ defined by J. Block and S. Weinberger

Uniformly finite homology

Block-Weinberger Theorem

For a finite complex K, $H_0^{uf}(\widetilde{K};\mathbb{Z}) = 0$ if and only if $\pi_1(K)$ is not amenable.

Counter-Example to Gromov's Conjecture

There is a closed rationally essential n-manifold M, $n \geq 5$, with the fundamental group $\pi_1(M) = \mathbb{Z}^n \times F_2$ such that $\dim_{MC} \widetilde{M} < n$.

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Proof

We note that $B\pi = T^n \times (S^1 \vee S^1)$ for $\pi = \pi_1(M)$ where T^n is the n-torus. Consider the natural inclusion of T^n into $B\pi$. Then the image of the fundamental class $[T^n]$ in $H_n(B\pi)$ is $[T^n] \otimes 1$ where $1 \in H_0(S^1 \vee S^1)$. By Remark and Block-Weinberger theorem, $pert_*^{F_2}(1) = 0$. Therefore,

$$pert_*^{\pi}([T^n] \otimes 1) = pert_*^{\mathbb{Z}^n}([T^n]) \otimes pert_*^{F_2}(1) = 0.$$

By a surgery in dimension 1 and 2 performed on the torus T^n we can obtain a manifold M together with a map $f : M \to B\pi$ inducing isomorphism of the fundamental groups and such that $f([M]) = [T^n] \otimes 1$. Then $\dim_{MC} \widetilde{M} < n$.