The Cheeger-Müller theorem and generalizations

Luiz Hartmann

Universidade Federal de São Carlos - UFSCar

February 21, 2013

This work was partially supported by Grant FAPESP 2008/57607-6.

FSU Topology Week

In 1935, Reidemeister defined the *torsion* invariants to a manifold who are not simply connected. The torsion defined to a cell (or simplicial) complex was created during the study of Lens spaces $L(p, q)$. If we fix p, such spaces have the same homology and homotopy groups. Then, a natural question about the classification of this spaces arises. Lens spaces was the first counter-example for a conjecture made by Hurewicz:

> "Two homotopy equivalent manifolds are homeomorphic."

This conjecture is true in dimension **two**, but false in higher dimensions. For example, in dimension **three**, $L(7,1)$ and $L(7,2)$ are homotopy equivalent but are not homeomorphic. This spaces were totally classify using the Reidemeister torsion. So the Reidemeister torsion can detect how the fundamental group act in the cellular(simplicial) structure of the space. J. Milnor wrote in (1966): "This torsion is a kind of determinant which describes the way in which the simplexes of $\tilde{X}$ are fitted together with respect to the action of the fundamental group ".

Reidemeister Torsion

Let V is a finite real vector space. If $v = \{v_1, \ldots, v_k\}$ and $w = \{w_1, \ldots, w_k\}$ are two basis of V, we denote by (w/v) the change of basis matrix from w to v. Let

$$C : \qquad C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

a finite chain complex of finite real vector spaces.
Denote by $Z_q = \ker \partial_q$, $B_q = \mathrm{Im}\partial_{q+1}$, e $H_q(C) = Z_q/B_q$ as usual. For each q, fix a basis $\mathbf{c}_q$ for C_q, and a basis $\mathbf{h}_q$ for $H_q(C)$. Let $\mathbf{b}_q$ be a set of (independent) elements of C_q such that $\partial_q(\mathbf{b}_q)$ is a basis for B_{q-1}. Then the set of elements $\{\partial_{q+1}(\mathbf{b}_{q+1}), \mathbf{h}_q, \mathbf{b_q}\}$ is a basis for C_q.

Definition (Reidemeister Torsion)

The Reidemeister torsion of the complex C with respect to the graded basis $\mathbf{h} = \{\mathbf{h}_q\}$ is the positive real number

$$\log \tau(C; \mathbf{h}) = \sum_{q=0}^{m} (-1)^q \log \left|\det\left(\partial_{q+1}(\mathbf{b}_{q+1}), \mathbf{h}_q, \mathbf{b}_q / \mathbf{c}_q\right)\right|,$$

Let W be a compact, orientable Riemannian manifold of dimension m possible with boundary ∂W. Let K be the triangulation C^k, $k > 2$, of W, which contain a subcomplex L, triangulating ∂W. Let $\tilde{K}$ be the universal recovering of K, and identify the fundamental group of K with the group of the covering transformations of $\tilde{K}$. We consider the complex $C(\tilde{K}, \tilde{L})$ a module over the group ring $\mathbb{R}\pi_1(K)$. A natural preferred basis is given by choosing cells covering those in $K - L$. Let $\rho : \pi_1(K) \to O(n, \mathbb{R})$ be a representation of the fundamental group in some group $O(n, \mathbb{R})$. Define the complex

$$C_q(K, L; \rho) = \mathbb{R}^n \otimes_{\mathbb{R}\pi_1(K)} C_q(\tilde{K}, \tilde{L}).$$

Then $C_q(K, L, \rho)$ is a finite real vector space and we can choose a preferred basis $e_j \otimes \mathbf{c}_q$ para $C_q(K, L; \rho)$, where $\mathbf{c}_q$ is the preferred basis of $C_q((\tilde{K}, \tilde{L}); \mathbb{R}\pi_1(K))$ and e_j are the elements of an orthonormal basis of $\mathbb{R}^k$.

Definition (Reidemeister torsion of a manifold)

The Reidemeister torsion of $(W, \partial W)$ *with the basis* $\mathbf{h} = \{\mathbf{h}_q\}$ *is*

$$\tau(W, \partial W; \mathbf{h}) = \tau(C(K, L; \rho); \mathbf{h}).$$

If $C(K, L; \rho)$ is acyclic, J. Minor proved that $\tau(W, \partial W)$ is a combinatorial invariant.

In 1971, D.B. Ray e I.M. Singer defined the Analytic torsion and conjectured the equality between both torsions.

For this, they fixed the homology basis of the complex as follows. Let W a compact oriented Riemannian m-dimensional manifold possible with boundary. First, consider an orthonormal basis of the harmonic forms of W, then applying the isomorphisms from De Rham, Poincaré and Hodge duality we obtain a basis $\mathbf{h}_q$ of H_q and so a basis $\partial_{q+1}(\mathbf{b}_{q+1}), \mathbf{h}_q, \mathbf{b}_q$ of $C_q(K, L; \rho)$. Note that, Ray and Singe proved that the Reidemeister Torsion, with this basis, is a combinatorial invariant.

Definition (Reidemeister torsion with the Ray and Singer basis)

The Reidemeister torsion with the Ray and Singer basis is defined by

$$\tau(W, \partial W) = \tau(C(K, L; \rho); \mathbf{a}_{\mathrm{rel}})$$

to the pair $(W, \partial W)$ *(relative case), and*

$$\tau(W) = \tau(C(K; \rho); \mathbf{a}_{\mathrm{abs}})$$

to W *(absolute case), where* $\mathbf{a}_{\mathrm{rel}}$ *(*$\mathbf{a}_{\mathrm{abs}}$*) is the image of an orthonormal basis from the harmonic forms of* W *with relative (absolute) boundary conditions in each* q*.*

Analytic Torsion

Consider W as previously and let g be the riemannian metric of W. Let $E_\rho \to W$ be the real vector bundle associated to the representation $\rho : \pi_1(W) \to O(n, \mathbb{R})$. Let $\Omega_{\text{abs}}(W, E_\rho)$ ($\Omega_{\text{rel}}(W, E_\rho)$) be the graded linear space of smooth forms on W with values in E_ρ and absolute boundary conditions (relative boundary conditions). We know that $\Omega_{\text{abs}}(W, E_\rho)$ is an Hilbert space. Let $d^\dagger = (-1)^{mq+m+1} * d*$ be the formal adjoint of d, then the Laplacian $\Delta_{\text{abs}} = (d^\dagger d + dd^\dagger)$ is a symmetric non negative definite operator in $\Omega_{\text{abs}}(W, E_\rho)$, and has pure point spectrum $\text{Sp}\Delta$. Let $\Delta_{\text{abs}}^{(q)}$ be the restriction of Δ_{abs} to $\Omega^q_{\text{abs}}(W, E_\rho)$.

Then we define the *zeta function* of $\Delta_{\text{abs}}^{(q)}$ by the series

$$\zeta(s, \Delta_{\text{abs}}^{(q)}) = \sum_{\lambda \in \text{Sp}_+ \Delta_{\text{abs}}^{(q)}} \lambda^{-s},$$

for $\text{Re}(s) > \frac{m}{2}$, and where Sp_+ denotes the positive part of the spectrum. The above series converges uniformly for $\text{Re}(s) > \frac{m}{2}$, and extends to a meromorphic function analytic at $s = 0$.

Definition

The analytic torsion of (W, g) is defined with respect to the representation ρ with by

$$\log T_{\text{abs}}(W; g) = \frac{1}{2} \sum_{q=1}^{m} (-1)^q q \zeta'(0, \Delta_{\text{abs}}^{(q)}).$$

Cheeger-Müller theorem

In 1978, Jeff Cheeger and Werner Müller proved independently the conjecture of Ray and Singer, in the case of a closed manifold.

Theorem (Cheeger-Müller theorem)

If W is a closed oriented Riemannian manifold, $T(W)$ is the Analytic Torsion and $\tau(W)$ is the Reidemeister Torsion then $T(W) = \tau(W)$.

1. Reidemeister Torsion
2. Analytic Torsion
3. Cheeger-Müller theorem
4. Generalizations to the Cheeger-Müller theorem
   - Manifolds with boundary
   - Manifold with conical singularities

After that the theory followed two natural ways:

1. A Cheeger-Müller type theorem to manifolds with boundary.
2. Extend the concept of Analytic Torsion and Reidemeister Torsion to manifolds with singularities and a Cheeger-Müller type theorem in this case.

First Case, manifolds with boundary

In the first case, W. Lück (1993) proved an extension to manifolds with boundary, but with a restriction that the metric of the manifold near the boundary has a product structure.

Theorem (W. Luck(1993))

Let W is a compact oriented Riemannian manifold with boundary ∂W. Assume that the metric is a product near the boundary and ρ is a representation of the fundamental group of W in $O(n, \mathbb{R})$. Then,

$$\log T_{\text{abs}}(W) = \log \tau(W) + \frac{\text{rk}(\rho)}{4} \chi(\partial W) \log 2.$$

More recently, the boundary term was study in general by J. Brüning and X. Ma (2006). They proved that

Theorem (Cheeger-Müller theorem to manifolds with boundary)

$$\log T_{\mathrm{abs}}(W) = \log \tau(W) + \frac{\mathrm{rk}(\rho)}{4}\chi(\partial W)\log 2 + \mathrm{rk}(\rho)A_{\mathrm{BM}}(\partial W).$$

The term $A_{\mathrm{BM}}(\partial W)$ will be called *anomaly boundary term.*

Definition of $A_{\mathrm{BM}}(\partial W)$

Recalling the splitting of the tangent bundle TW on the collar $C(\partial W) = [0,\epsilon) \times \partial W$, we define the section s_j of one forms with values in the skew-adjoint endomorphisms of TW

$$s_j = \nabla_j^{TW} - \nabla^{TC(\partial W)} = (\nabla_j^{TW} P_{\mathrm{tg}})_{\mathrm{norm}} + (\nabla_j^{TW} P_{\mathrm{norm}})_{\mathrm{tg}},$$

where ∇_j^{TW} is the Levi-Civita connection on (W, g_j), $\nabla^{TC(\partial W)}$ is the Levi-Civita connection on $C(\partial W)$, and P denotes the projection associated to the splitting $TC(\partial W) = N\partial W \otimes T\partial W$.

We set

$$\mathcal{S}_j = \frac{1}{2}\sum_{k=1}^{m-1}(i^* s_j(e_k))_{\text{norm}}\hat{e}^k,$$

$$\mathcal{R} = \hat{\hat{R}} = \frac{1}{2}\sum_{j,l=1}^{n}(e_j, \tilde{R}e_l)\hat{e}^j \wedge \hat{e}^l,$$

where $(e_{y_1}, \dots, e_{y_{m-1}})$ is an orthonormal basis in $T\partial W$, $(e^{y_1}, \dots, e^{y_{m-1}})$ is the dual basis of $(e_{y_1}, \dots, e_{y_{m-1}})$ and $\tilde{R}$ is the 2-form curvature of ∂W.

Definition

$$B(\nabla_j^{TW}) = \frac{1}{2}\int_0^1 \int^B \mathrm{e}^{-\frac{1}{2}\mathcal{R} - u^2\mathcal{S}_j^2}\sum_{k=1}^{\infty}\frac{1}{\Gamma\left(\frac{k}{2}+1\right)}u^{k-1}\mathcal{S}_j^k du.$$

So the anomaly boundary term $A_{\mathrm{BM}}(\partial W)$ is described by the following theorem,

Theorem (Analytic torsion comparation)

Let W be a compact oriented Riemannian manifold with boundary ∂W, with two metrics g_0 and g_1, ρ is a representation of $\pi_1(W)$ in $O(n, \mathbb{R})$. Then,

$$A_{\mathrm{BM}}(\partial W) = \log \frac{T_{\mathrm{abs}}(W, g_1)}{T_{\mathrm{abs}}(W, g_0)} = \frac{\mathrm{rk}(\rho)}{2} \int_{\partial W} \left(B(\nabla_1^{TW}) - B(\nabla_0^{TW}) \right),$$

where ∇_j^{TW} is the Levi-Civita connection of the metric g_j.

Gluing formulas

It is very important to obtain gluing formulas for torsions as, for example, this formulas can be used to facilitate the calculations of such torsions
For the Reidemeister torsion, the proof of this type of formula is algebraic and it was presented by J. Minor.

Theorem

Let M a closed Riemannian manifold such that $M = M_1 \cup_W M_2$, where M_1 and M_2 are compact manifolds with boundary $\partial M_1 = \partial M_2 = W$. Then,

$$\log \tau(M) = \log \tau(M_1) + \log \tau(M_2, W) + \log \tau(\mathcal{H}),$$

where $\mathcal{H}$ is the exact homology sequence of the pair (M, M_1).

For the Analytic torsion, in 1995, S.M. Vishik proved the following gluing formula,

Theorem

Let M be a closed Riemannian manifold such that $M = M_1 \cup_W M_2$, where M_1 and M_2 are compact Riemannian manifolds with boundary $\partial M_1 = \partial M_2 = W$, and the metric of M is product near the gluing. Then,

$$\log T(M) = \log T(M_1) + \log T(M_2, W) + \log T(\mathcal{H}) + \frac{1}{2}\chi(W) \log 2,$$

where $\mathcal{H}$ is the exact sequence of the pair (M, M_1), in the harmonic forms.

In 2012, J. Brüning and X. Ma extended the previously theorem without the assumption that the metric is a product near the gluing, and the result is:

Theorem

Let M be a closed Riemannian manifold such that $M = M_1 \cup_W M_2$, where M_1 and M_2 are compact Riemannian manifolds with boundary $\partial M_1 = \partial M_2 = W$. Then,

$$\log T(M) = \log T(M_1) + \log T(M_2, W) + \log T(\mathcal{H}) + \frac{1}{2}\chi(W)\log 2 + (-1)^{\dim M} A_{\mathrm{BM}}(W),$$

where $\mathcal{H}$ is the exact sequence of the pair (M, M_1), in the harmonic forms.

Second Case, pseudomanifold with conical singularities

In the second case, Jeff Cheeger constructed the bases for the extension of the Analytic torsion of limited metric cone,

Definition

The limited metric cone $C_l N$, where N is a closed Riemannian manifold is the space $[0, l] \times N$ with the metric $g = dr \otimes dr + r^2 g_N$ to $r > 0$.

Definition

X^{m+1} is a pseudomanifold with conical singularities if exist $p_j \in X^{m+1}$, $j = 1, \ldots, k$, such that $X^{m+1} - \cup_{j=1}^{k} p_j$ is an open Riemannian manifold (possible with boundary) and each p_j has a neighborhood U_j such that $U_j - \{p_j\}$ is isometric to a $C_{l_j} N_j^m$ for some $l_j \in \mathbb{R}$.

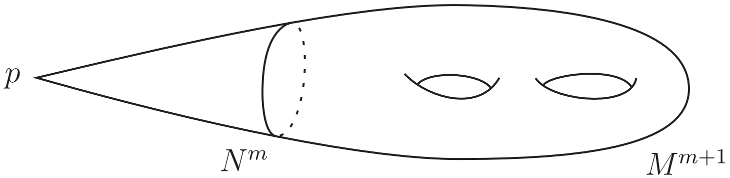

Figure : Example of a pseudomanifold with conical singularities.

The extension of the definition of Analytic torsion and the Reidemeister torsion to a pseudomanifold with conical singularities was made by A. Dar, in 1987.

Definition (Intersection Reidemeister torsion)

$$\log I\tau(X) = \frac{1}{2}\left(\log I^{m}\tau(X) + \log I^{m^c}(X)\right)$$

Definition (Analytic torsion of a pseudomanifold with conical singularities)

$$\log T(X) = \frac{1}{2}\sum_{q=1}^{\dim X}(-1)^q q\zeta'(0, \Delta^{(q)}_{L^2}).$$

Extension of Cheger-Müller theorem to singular cases

The first result in this direction is:

Theorem (Extension of Cheger-Müller theorem to a cone over a odd dimensional sphere)

The Analytic torsion of a cone over a odd dimensional sphere is

$$\log T(C_l S^{2p-1}_{\sin\alpha}) = \log I\tau(C_l S^{2p-1}_{\sin\alpha}) + A_{\mathrm{BM}}(S^{2p-1}_{l\sin\alpha}).$$

Note that, we don't have contribution from the singularity.

Proposition

Let N a closed Riemannian manifold. Then the boundary contribution in the Analytic torsion of $C_l N$ coincide with the boundary contribution of the smooth case.

Theorem (Cheeger-Müller theorem of an even dimension limited metric cone)

If W a closed Riemanian odd dimensional manifold, then

$$\log T(C_l N) = \log I\tau(C_l N) + A_{\mathrm{BM,abs}}(\partial C_l N).$$

The even case

Consider

$$A_{0,0,q}(s) = \sum_{n=1}^{\infty} \left(\log\left(1 - \frac{\alpha_q}{\mu_{q,n}}\right) - \log\left(1 + \frac{\alpha_q}{\mu_{q,n}}\right)\right) \frac{m_{q,n}}{\mu_{q,n}^{2s}},$$

where $\alpha_q = \frac{2q+1-2p}{2}$, $\mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2}$ and $m_{q,n}$ is the multiplicity of the eigenvalor $\lambda_{q,n}$ of W.

Theorem (Hartmann-Spreafico-Vertman)

Let (W, g) a closed Riemanian even dimensional manifold. Then,

$$\log T_{\mathrm{abs}}(C_l W) = \frac{1}{2}\left(\sum_{q=0}^{p-1}(-1)^q r_q \log \frac{l^{2p-2q+1}}{2p-2q+1} + (-1)^p \frac{r_p}{2}\log l\right)$$

$$+ \frac{1}{2}\chi(W)\log 2 + \sum_{q=0}^{p-1}(-1)^{q+1} r_q \log(2p-2q-1)!!$$

$$+ \frac{1}{2}\sum_{q=0}^{p-1}(-1)^{q+1} A_{0,0,q}(0) + A_{\mathrm{BM}}(W).$$

Gluing Formulas - Intersection Torsion

For the Intersection Reidemeister Torsion, the proof of J. Milnor holds in this case too. Then we have,

Theorem

Let M a closed pseudomanifold with conical singularities such that $M = M_1 \cup_W M_2$, where M_1 and M_2 are compact pseudomanifold with conical singularities with boundary $\partial M_1 = \partial M_2 = W$. Then,

$$\log I\tau(M) = \log I\tau(M_1) + \log I\tau(M_2, W) + \log I\tau(\mathcal{H}),$$

where $\log I\tau(\mathcal{H})$ is the average of the torsion of the exact intersection homology sequece of the (M, M_1) with lower middle perversity and upper middle perversity.

Gluing Formulas - Analytic Torsion

For the Analytic torsion M. Lesch extended the result of M. Vishik:

Theorem

Let M a closed pseudomanifold with conical singularities such that $M = M_1 \cup_W M_2$, where M_1 and M_2 are compact pseudomanifold with conical singularities with boundary $\partial M_1 = \partial M_2 = W$, and the metric of M is a product near the gluing. Then,

$$\log T(M) = \log T(M_1) + \log T(M_2, W) + \log T(\mathcal{H}) + \frac{1}{2}\chi(W)\log 2,$$

where $\mathcal{H}$ is the exact sequence of the pair (M, M_1), in the harmonic forms.

What we need to understand the complete extension?

- To understand the geometric contribution of the Theorem 10;
- A gluing formula without the metric restriction near the gluing (Theorem 12).

Thank you!