Some invariants on coincidences of fiberwise maps

Alice Kimie Miwa Libardi

IGCE, São Paulo State University (Unesp) - Brazil

FSU Topology Week
February, 2013

In this talk I will present some results of *Coincidences of fiberwise maps between sphere bundles over the circle*, joint work with D. Gonçalves, U. Koschorke and O. Neto.

Let $A_M : F_M \to F_M$ and $A_N : F_N \to F_N$ be self diffeomorphisms of smooth, closed, connected manifolds F_M and F_N (of dimensions $m-1$ and $n-1$, resp.).

Consider the fiberwise maps

$$\begin{array}{ccc} f, f_1, f_2, \ldots : M := (I \times F_M)/((1,x) \sim (0, A_M(x))) & \longrightarrow & N := (I \times F_N)/((1,y) \sim (0, A_N(y))) \\ \quad\searrow^{p_M} & & {}^{p_N}\swarrow \quad \\ & S^1 = I/(1 \sim 0) & \end{array}$$

Question I

Can the coincidence locus

$$C(f_1, f_2) = \{x \in M, f_1(x) = f_2(x)\}$$

be made empty by suitable fiberwise homotopies of f_1 and f_2?

If this can be done, we say that (f_1, f_2) is loose over S^1.

If **Question I** has no positive answer, we want to measure somehow to what extend the pair (f_1, f_2) fails to be loose over S^1.

This can be done, via the minimum number of coincidences path-components

$$MCC_{S^1}(f_1, f_2) = \min\{\sharp\pi_0(C(f_1', f_2')), f_1' \sim_{S^1} f_1,\ f_2' \sim_{S^1} f_2\}.$$

Clearly, $MCC_{S^1}(f_1, f_2) = 0 \Leftrightarrow (f_1, f_2)$ is loose.

Question II

$MCC_{S^1}(f_1, f_2) = N_{S^1}(f_1, f_2)$, for all fiberwise maps $f_1, f_2 : M \to N$?

Question III

Can we classify the fiberwise homotopy classes of fiberwise maps, or at least, get some bounds for their number?

Let X be a topological space, $\phi = \phi^+ - \phi^-$ be a virtual bundle over X. An element of the nth normal bordism group of X, with coefficient ϕ, denoted by $\Omega_n(X;\phi)$ is a class $[(K^n, g, \bar{g})]$ where $[K^n, g]$ is a bordism class (in the Conner–Floyd sense) and $\bar{g}$ is a stable vector bundle isomorphism:

$$\bar{g} : TK \oplus g^*(\phi^+) \to g^*(\phi^-)$$

General case

Consider $f_1, f_2 : M \to N$ fiberwise maps over B.

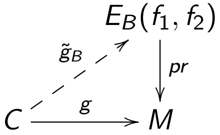
$$(f_1, f_2) : M \to N \times_B N$$

and Δ the diagonal of $N \times N$.
After small deformation of f_1 and f_2 over B, (f_1, f_2) is transversal to Δ and the coincidence locus $C = C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta)$. So C is an $(m - n + b)$ dimensional submanifold of M with a vector bundle isomorphism

$$\bar{g}_B : \nu(C, M) = (f_1, f_2)\big|_C^*(\nu(\Delta, N \times_B N)) = f_1^*(TF(p_N))\big|_C.$$

Here $TF(p_N)$ denotes the tangent bundle along the fibers of p_N.

Also consider



where

$$E_B(f_1, f_2) := \{(x, \theta) \in M \times N^I, p_N\theta \equiv p_M(x), \theta(0) = f_1(x), \theta(1) = f_2(x)\}$$

and $\tilde{g}_B(x) = (x, \text{cte path at } f_1(x) = f_2(x))$.

Then we define $\omega_B(f_1, f_2) = [(C, \tilde{g}_B, \bar{g}_B^\sharp)] \in \Omega_{m-n+b}(E_B(f_1, f_2); \varphi)$ where

$$\bar{g}_B^\sharp : TC \oplus f_1^*(TF(p_N))\big|_C \to TM$$

The class $\omega_B(f_1, f_2)$ is the strongest obstruction for coincidences of fiberwise maps, but it is difficult to calculate.
There is a weaker invariant, that is, the stabilized version

$$\begin{aligned}\tilde{\omega}_B(f_1, f_2) &= [(C, \tilde{g}_B, \bar{g}_B)] \in \Omega_{m-n+b}(E_B(f_1, f_2); \varphi) \\ &= \bigoplus_{A \in \pi_0(E_B(f_1, f_2))} \Omega_{m-n+b}(A, \varphi|_A)\end{aligned}$$

where

$$\bar{g}_B : TC \oplus \tilde{g}_B^*(f_1^*(TF(p_N)) \oplus \epsilon^k \simeq \tilde{g}_B^*(TM) \oplus \epsilon^k, k \gg 0$$

and $\varphi := pr^*(f_1^*(TF(p_N)) - TM)$.
A path component A of $E_B(f_1, f_2)$ is called essential if the correspondent direct summand of $\tilde{\omega}_B(f_1, f_2)$ is nontrivial.
The Nielsen coincidence number $N(f_1, f_2)$ is the number of essential path components $A \in \pi_0(E_B(f_1, f_2))$.

Theorem [K,G]: Assume that $m < 2(n - b) - 2$. Then a pair (f_1, f_2) is loose over B if and only if $\tilde{\omega}_B(f_1, f_2) = 0$.

In our case $B = S^1$, $(f_1, f_2) : M \to N \times_{S^1} N$, $C = (f_1, f_2)^{-1}(\Delta)$ is an $(m - n + 1)$ submanifold of M and the invariant $\omega_{S^1}(f_1, f_2) \in \Omega_{m-n+1}(M; \varphi)$, where φ is trivial or $\varphi = p_M^*(\lambda)$, where λ denotes the nontrivial line bundle over S^1.

From now on we consider A_M and A_N orthogonal self diffeomorphisms and we assume A_M of the form

$$A_M(x_1, \dots, x_m) = (x_1, \dots, x_{m-1}, \pm x_m)$$

for all $(x_1, \dots, x_m) \in \mathbb{R}^m$ and similarly $A_N : \mathbb{R}^n \to \mathbb{R}^n$.

Both gluing maps leave the point $* = (1, 0, \dots, 0)$ fixed. Thus we get zero-sections s_{0M} in M and s_{0N} in N described by $[t] \in S^1 \mapsto [t, *]$ as well as the fiberwise maps $f_0 = s_{0N} \circ p_M : M \to N, f_0(x) = *$ and $a \circ f_0 : M \to N$, $(a \circ f_0)(x) = a(*)$, where a denotes the antipodal map.

Let us consider

$$\mathcal{F} := \{f : M \to N \text{ fiberwise maps}\}/\text{fiberwise homotopy.}$$

The second invariant is defined as

$$\deg_{S^1} : \mathcal{F} \to \Omega_{m-n+1}(M;\varphi)$$
$$\deg_{S^1}([f]) = \omega_{S^1}(f, a \circ f_0)$$

For $m, n \geq 2$ the group $\Omega_{m-n+1}(M; \varphi)$ fits into a long exact Gysin sequence

$$\cdots \longrightarrow \Pi_i^S \oplus \Pi_{i-m+1}^S \xrightarrow{\text{incl}_*} \Omega_i(M; \varphi) \xrightarrow{\pitchfork} \Pi_{i-1}^S \oplus \Pi_{i-m}^S \longrightarrow \cdots$$

where Π_i^S is the stable homotopy groups of spheres.
Given an element of $\Omega_i(M; \varphi)$ we may represent it by a singular manifold (C, c), $c : C \to M$, c smooth and transverse to the fiber $F_M \subset M$.
Then the intersection manifold $c^{-1}(F_M)$ together with the stable framing $\bar{c}$ induced from $c| : c^{-1}(F_M) \to F_M$, represents $\pitchfork([C, c, \bar{c}])$ where $\bar{c}$ is a vector bundle isomorphism relating the stable normal bundle of C to $c^*(\varphi)$.
In particular, $\forall \ [f_1], [f_2] \in \mathcal{F}$,

$$\pitchfork(\omega_{S^1}(f_1, f_2)) = \omega(f_1|, f_2|)$$

where $f_1|$ and $f_2| : F_M \to F_N$.

If $n = 2$ we have a weaker invariant which in some cases it is still a complete looseness obstruction $\mu(\omega_{S^1}(f_1, f_2))$ where

$$\mu : \Omega_{m-1}(M; \varphi) \to H_{m-1}(M; \widetilde{\mathbb{Z}}_\varphi)$$

is the generalized Hurewicz homomorphism.

Theorem: For $n \geq 3$, the following conditions are equivalent:

1. $\forall \, [f_1], [f_2] \in \mathcal{F}$ we have

$$\omega_{S^1}(f_1, f_2) = 0 \Leftrightarrow (f_1, f_2) \text{ is loose over } S^1$$

2. $MCC_{S^1}(f_1, f_2) = N_{S^1}(f_1, f_2), \forall \, [f_1], [f_2] \in \mathcal{F}$
3. The map $\deg_{S^1} : \mathcal{F} \to \Omega_{m-n+1}(M; \varphi)$ is injective

The proof uses the following diagram, $n \geq 3$

$$
\begin{array}{ccccccccc}
 & & \operatorname{coker}((\delta_m - \mathrm{id}) : \pi_m(S^{n-1}) \circlearrowleft) & \overset{\mathrm{act}}{\dashrightarrow} & \mathcal{F} & \xrightarrow{q} & \ker((\delta_{m-1} - \mathrm{id}) : \pi_{m-1}(S^{n-1}) \circlearrowleft) & & \\
 & & \downarrow{\scriptstyle E_2^\infty} & & \downarrow{\scriptstyle \deg_B} & & \downarrow{\scriptstyle E_1^\infty} & & \\
0 & \longrightarrow & \operatorname{coker}((d-1)\cdot : \pi^S_{m-n+1} \circlearrowleft) & \xrightarrow{\mathrm{incl}_*} & \Omega_{m-n+1}(M,\varphi) & \xrightarrow{\pitchfork} & \ker((d-1)\cdot : \pi^S_{m-n} \circlearrowleft) & \longrightarrow & 0
\end{array}
$$

Here the lower horizontal sequence is exact and comes from Gysin and Pontriagin–Thom isomorphism.

E_1^∞ and E_2^∞ denote stable suspension homomorphism.

q is the restriction to the fiber $F_M = S^{m-1}$ over $[1] = [0] \in S^1$.

$d_M = \det A_M; \quad d_N = \det A_N; \quad d = d_M d_N.$

$\delta_i : \pi_i(S^{n-1}) \to \pi_i(S^{n-1})$, $\delta_i([u]) = d_M (A_N)_*([u])$.

$m = n = 3$, $d_M = d_N = 1$, $\delta_3 = \delta_2 = id$, $d = 1$,
E_1^∞ is an isomorphism and $\deg_{S^1}$ is onto.

$n \geq 3$. If E_1^∞ and E_2^∞ are injective (or surjective, resp.) then $\deg_{S^1}$ is also injective (or surjective).

$\deg_{S^1}$ is an isomorphism in the stable dimension range $m \geq 2n - 4$.

- D. L. Gonçalves and U. Koschorke, *Nielsen coincidence theory of fibre-preserving maps and Dold's fixed point index*, Top. Math. in Nonlinear Analysis, **33** (2009), 85–103.
- D. Gonçalves, U. Koschorke, A. Libardi and O. Neto, *Coincidences of fiberwise maps between sphere bundles over the circle*, (to appear in Proc. Edinburgh Math. Soc.)