# *Results on local Nash equilibrium*


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*joint work with Carlos Biasi*

# *Presentation*

# *Introduction*

In the paper *"Weak local Nash equilibrium"* we define a concept of local Nash equilibrium and we prove its existence applying the Lefschetz fixed point theorem.

The concept of Nash equilibrium says that an equilibrium to the functions

$$p_1, p_2, \dots, p_n : S = S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

is a point $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \in S$ such that, for each $i \in \{1, 2, \dots, n\}$,

$$p_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \dots, \tilde{s}_n) \leq p_i(\tilde{s}), \text{ for all } s_i \in S_i.$$

As an application of Brouwer fixed point theorem, Nash proved that:

*Theorem (Nash)*

*Let $S_1, \dots, S_n$ be compact convex subsets of an Euclidean space. Suppose that $p_1, \dots, p_n : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ are maps such that, for each $i = 1, \dots, n$, $p_i(s_1, \dots, s_n)$ is linear(afim) as a function of $s_i$. Then there exists at least one equilibrium to $p_1, \dots, p_n$ .*

*Proof:*

$$p_i(s_1, \dots, s_n) = \langle v_i(s_1, \dots, s_n), s_i \rangle + u_i(s_1, \dots, s_n),$$

where the maps $v_i : S_1 \times \cdots \times S_n \to \mathbb{R}^{d_i}$ and $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ do not depend on $s_i$.

$r : \mathbb{R}^m \to S$ the natural retraction.

$f : S \to S$ by $f(s) = r(s + v(s))$, where $v(s) = (v_1(s), \dots, v_n(n))$

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Based on the proof of Nash's Theorem, we investigated the existence of equilibrium in the context that the spaces of strategies are compact ENR's, not necessarily convex. From this study, we define:

*Definition*

*Let* $(S_1, d_1), \ldots, (S_n, d_n)$ *be metric spaces and* $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ *real functions. We say that* $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in S$ *is a* **weak local equilibrium (abbrev., w.l.e.)** *for* $p_1, \ldots, p_n$ *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon d_i(s_i, \tilde{s}_i),$$

*for every* $s_i \in B(\tilde{s}_i, \delta)$*,* $i = 1, 2, \ldots, n$*.*

In the previous paper, we were able to prove the following result.

*Theorem*

*Let $p_1, \ldots, p_n : S_1 \times \ldots \times S_n \to \mathbb{R}$ be maps, where each $S_i \subset \mathbb{R}^{m_i}$ is a compact ENR with the p.c.r.. Also, suppose $p_i(s_1, \ldots, s_n)$ continuously differentiable in a neighborhood of $s_i$ when the other variables are kept fixed, $i = 1, 2, \ldots, n$. If $\chi(S_i) \neq 0$ for $i = 1, 2, \ldots, n$ then $p_1, p_2, \ldots, p_n$ have at least one w.l.e..*

# *Definitions*

The open ball in $\mathbb{R}^n$ with center in $x_0$ and radius $r > 0$ will be denoted by $B(x_0, r)$.

*Definition*

*Let $f : A \to \mathbb{R}$ be a function, where $A$ is an open non-empty subset of $\mathbb{R}^n$. Given $x_0 \in A$, we say that $f$ is upper semi differentiable(u.s.d.) at $x_0$ if there exists at least one point $v \in \mathbb{R}^n$ with a function $r : B(0, \varepsilon) \to \mathbb{R}$ such that $\lim_{h \to 0} \frac{r(h)}{\|h\|} = 0$ and*

$$f(x_0 + h) \leq f(x_0) + v \cdot h + r(h)$$

*for every $h$ such that $x_0 + h \in A$.*

We denote by $DSf(x_0)$ the set of such vectors $v$.

# Some Properties

*Example*

*If $f : A \to \mathbb{R}$ is differentiable at $x_0$ then $f$ is u.s.d. and, in this case, $DSf(x_0) = \{f'(x_0)\}$.*

*Theorem*

*If $f$ is u.s.d. at $x_0$ then $DSf(x_0)$ is convex subset of $\mathbb{R}^n$.*

*Theorem*

*Let $f : J \to \mathbb{R}$ be a function, where $J \subset \mathbb{R}$ is open interval, and let $x_0 \in J$. Suppose the existence of the right and left-hand limits*

$$c = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$$

*and*

$$d = \lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

*Then, $f$ is u.s.d. if and only if $c \leq d$. Moreover, $DSf(x_0) = [c, d]$.*

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*Example*

$$f(x) = \begin{cases} x, & \text{if } x < 0 \\ -x, & \text{if } x \geq 0 \end{cases}$$

*is u.s.d. And,* $DSf(x) = \{1\}$ *if* $x < 0$, *DSf(x)={-1} if* $x > 0$, $DSf(0) = [-1, 1]$.

*Notation*

Let $f : A \to \mathbb{R}$ be a map, where $A$ is an open subset of $\mathbb{R}^n$. Let $x_0 \in A$. We denote the right-hand partial derivatives and the left-hand partial derivatives, respectively, by

$$\frac{\partial f^+}{\partial x_i}(x_0) = \lim_{t \to 0^+} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

and

$$\frac{\partial f^-}{\partial x_i}(x_0) = \lim_{t \to 0^-} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

$i = 1, \ldots, n$

*Theorem*

*Let $f : A \to \mathbb{R}$ be a map, $A \subset \mathbb{R}^n$ open. Suppose well defined the right-hand and the left-hand partial derivatives of $f$ at every $x_0 \in A$. Also, suppose the functions*

$$\frac{\partial f^+}{\partial x_i}, \frac{\partial f^-}{\partial x_i} : A \to \mathbb{R}$$

*continuous and that*

$$\frac{\partial f^+}{\partial x_i}(x_0) \leq \frac{\partial f^-}{\partial x_i}(x_0), \ \forall \ x_0 \in A,$$

*$i = 1, \dots, n$. Then, $f$ is u.s.d. and*

$$DSf(x_0) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

*where $a_i = \dfrac{\partial f^+}{\partial x_i}(x_0)$, $b_i = \dfrac{\partial f^-}{\partial x_i}(x_0)$, $i = 1, \dots, n.$ , Thus, $DSf : A \multimap \mathbb{R}^n$ is an u.s.c. multivalued map with convex compact values $^{(*)}$.*

$(*)$ Let $X$ and $Y$ be two spaces and assume that for each point $x \in X$ a nonempty closed subset $\varphi(x)$ of Y is given; in this case, we say that $\varphi$ is a multi-valued map from $X$ into $Y$ and we write $\varphi : X \multimap Y$.

A multi-valued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) if for every open subset $U$ of $Y$ the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is an open subset of $X$.

# *ENR's with the property of convenient retraction*

*Definition*

*We say that a subset $X$ of $\mathbb{R}^m$ has the* **property of convenient retraction (abbrev., p.c.r.)** *if there exists a retraction $r : V \to X$, where $V$ is an open neighborhood of $X$ in $\mathbb{R}^m$, satisfying: given $x_0 \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\langle x_0 - r(x_0), x - r(x_0)\rangle \leq \varepsilon \|x - r(x_0)\|,$$

*for all $x \in X$ with $\|x - r(x_0)\| < \delta$, where $\langle \ , \ \rangle$ is the inner product in $\mathbb{R}^m$ and $\| \cdot \|$ is the norm induced by it. In this case, we say that $r : V \to X$ is a convenient retraction.*

## *Examples*

- Every closed convex subset $K$ of $\mathbb{R}^m$ has the p.c.r..
- Every submanifold $M$ of $\mathbb{R}^n$, of class $C^2$, with or without boundary, has the p.c.r..
- Let $X$ be a closed subset of $\mathbb{R}^n$ and let $V$ be an open neighborhood of $X$ in $\mathbb{R}^n$. We say that a continuous function $r : V \to X$ is a proximative retraction (or metric projection) if

$$||r(y) - y|| = \text{dist}(y, X), \quad \forall \; y \in V.$$

A compact subset $X$ of $\mathbb{R}^n$ is called a proximative neighborhood retract (PANR) if there exists a proximative retraction $r : V \to X$.
Every proximative retraction is a convenient retraction.

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# *The Main Theorem*

Now, we are able to study existence of w.l.e. in the following setting:

- $S_i \subset \mathbb{R}^{m_i}$ compact ENR with the p.c.r., $i = 1, \ldots, n$.
- $p_i : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ continuous, $i = 1, \ldots, n$.
- $p_i(s_1, \ldots, s_i, \ldots, s_n)$ as a function of $s_i = (s_1^1, \ldots, s_1^{m_i})$ satisfies:
  - $S_i \subset V_i \to \mathbb{R}$, $x_i \in V_i \longmapsto p_i(s_{-i}, x_i)$, where $(s_{-i}, x_i) = (s_1, \ldots, s_{i-1}, x_i, s_{i+1}, \ldots, s_n)$
  - $p_i(s_{-i}, \_) : V_i \to \mathbb{R}$ has the lateral partial derivatives
  - $$\frac{\partial p_i^{+}}{\partial x_i^j}(s_{-i}, \_), \frac{\partial p_i^{-}}{\partial x_i^j}(s_{-i}, \_) : V_i \to \mathbb{R}$$
    continuous, $j = 1, \ldots, m_i$
  - $$\frac{\partial p_i^{+}}{\partial x_i^j}(s_{-i}, x_i) \leq \frac{\partial p_i^{-}}{\partial x_i^j}(s_{-i}, \_), \ \forall\, x_i \in V_i$$

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With these assumptions, there is a multivalued map $F : S \multimap S$ u.s.c. with compact values such that if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s}$ is a w.l.e. to $p_1, \dots, p_n$. The construction of the multivalued map $F$ is given by the following:

- Since $S_1 \subset \mathbb{R}^{m_1}, \dots, S_n \subset \mathbb{R}^{m_n}$ are compact ENR's with the p.c.r., $S = S_1 \times \cdots \times S_n \subset \mathbb{R}^m$ is also, $m = m_1 + \cdots + m_n$. Thus, let $r : V \to S$ be a convenient retraction.
- Let $V : S \multimap \mathbb{R}^m$ be the multivalued map defined by

$$V(s) = V_1(s) \times \cdots \times V_n(s)$$

  where $V_i(s) = DSp_i(s_{-i}, s_i)$.
- There exists $t_1 > 0$ such that $s + tv \in V$ for all $s \in S$, $t \in [0, t_1]$, $v \in V(s)$

With these assumptions, there is a multivalued map $F : S \multimap S$ u.s.c. with compact values such that if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s}$ is a w.l.e. to $p_1, \ldots, p_n$. The construction of the multivalued map $F$ is given by the following:

- Since $S_1 \subset \mathbb{R}^{m_1}, \ldots, S_n \subset \mathbb{R}^{m_n}$ are compact ENR's with the p.c.r., $S = S_1 \times \cdots \times S_n \subset \mathbb{R}^m$ is also, $m = m_1 + \cdots + m_n$. Thus, let $r : V \to S$ be a convenient retraction.
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Finally, define $F : S \multimap S$ by

$$F(s) = \{r(s + t_1 v) \mid v \in V(s)\}$$

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*If $\tilde{s} \in F(\tilde{s})$ then $\tilde{s}$ is a w.l.e. to $p_1, \ldots, p_n$.*

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*If $\chi(S_i) \neq 0$, $i = 1, \ldots, n$, then $F$ has at least one fixed point. In particular, $p_1, \ldots, p_n$ have at least one w.l.e.*

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# *The Lefschetz Fixed Point Theorem for Admissible Multivalued Mappings*

From now on, the spaces considered are metric spaces. We are considering the Čech homology functor with compact carriers and with coefficients in $\mathbb{Q}$.

- A proper map $f : X \to Y$ means: $\forall\, K \subset X$ compact, $f^{-1}(K)$ is compact.
- A compact space $X$ is called acyclic if $H_0(X) = \mathbb{Q}$ and $H_q(X) = 0$ for $q > 0$.
- $p : (X, X_0) \to (Y, Y_0)$ is called a Vietoris map if $p : X \to Y$ is proper, $p^{-1}(Y_0) = X_0$ and $p^{-1}(y)$ is acyclic, for every $y \in Y$. Symbol: $p : (X, X_0) \Rightarrow (Y, Y_0)$

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*Theorem (Vietoris Mapping Theorem)*

*If $p : (X, X_0) \Rightarrow (Y, Y_0)$ is a Vietoris map then $p_* : H_*(X, X_0) \to H_*(Y, Y_0)$ is an isomorphism.*

*Definition*

*An u.s.c. multivalued map $\varphi : X \multimap Y$ is called acyclic if for every $x \in X$ the set $\varphi(x)$ is an acyclic subset of $Y$.*

*Definition*

*A multivalued map $\varphi : X \multimap Y$ is called admissible if there exists a space $\Gamma$ and mappings $p : \Gamma \Rightarrow X$, $q : \Gamma \to Y$ such that:*

- *$p$ is a Vietoris map,*
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*Let $\varphi : X \multimap Y$ be an admissible multivalued map. The set $\{\varphi\}_*$ of linear induced mappings is defined by*

$$\{\varphi\}_* = \{q_* p_*^{-1} : H_*(X) \to H_*(Y) \mid (p, q) \subset \varphi\}$$

*Definition*

*Two admissible multivalued maps $\varphi, \psi : X \multimap Y$ are called homotopic (written $\varphi \sim \psi$) if there exists an admissible multivalued map $\chi : X \times [0, 1]$ such that:*

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*Theorem*

*Let $\varphi : X \multimap Y$ be two admissible multivalued maps. Then $\varphi \sim \psi$ implies that there exists selected pairs $(p, q) \subset \varphi$ and $(\bar{p}, \bar{q}) \subset \psi$ such that*

$$q_* p_*^{-1} = \bar{q}_* \bar{p}_*^{-1}$$

Let $X$ be a compact ANR and let $\varphi : X \multimap X$ be an admissible multivalued map. Then, it is well defined the Lefschetz set $\Lambda(\varphi)$ of $\varphi$ by putting

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1}) = \sum_i (-1)^i \mathrm{trace}_i(q_* p_*^{-1}) \mid (p, q) \subset \varphi\}$$

*Theorem (Lefschetz fixed point theorem for admissible multivalued mappings)*

*Let $X$ be a compact ANR and $\varphi : X \multimap X$ be a compact admissible multivalued map. If $\Lambda(\varphi) \neq \{0\}$ then $Fix(\varphi) \neq \emptyset$.*

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Obrigada!
(Thank you!)