Bourgin-Yang Theorem for $G$-equivariant maps

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Joint work with
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**INTRODUCTION**

The classical Borsuk-Ulam theorem says that: \( \forall \) continuous map \( f \) from the sphere \( S(\mathbb{R}^{m+1}) \) into \( \mathbb{R}^m \),

\[ \exists x \in S(\mathbb{R}^{m+1}) \text{ such that } f(x) = f(-x). \]

Remark. Let us observe that if \( f : S(\mathbb{R}^m) \rightarrow \mathbb{R}^n \) is a continuous map and

\[ A(f) = \{ x \in S(\mathbb{R}^m) \mid f(x) = f(-x) \}, \]

we can define an equivariant map

\[ h : S(\mathbb{R}^m) \rightarrow \mathbb{R}^n \]

\[ x \mapsto f(x) - f(-x) \]

for which

\[ Z_h = h^{-1}(0) = \{ x \in S(\mathbb{R}^m) \mid h(x) = f(x) - f(-x) = 0 \} = A(f). \]

Then, to give an estimate for a covering dimension of \( A(f) \), it is enough to give an estimate for a covering dimension of \( Z_h \).
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**Bourgin-Yang Theorem** Let \( f : S(\mathbb{R}^m) \to \mathbb{R}^n \) be a equivariant map. Then,

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\dim Z_f \geq m - n - 1
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To prove this theorem it is enough to show that \( H^{m-n-1}(Z_f/Z_2) \neq 0 \), which implies \( \dim Z_f \geq m - n - 1 \).

The situation for \( Z_p \), \( p \) prime, is analogous. [Munkholm, Izydorek & Rybicki, de Mattos & dos Santos, ...]
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We study the B-Y problem for $G$, as follows.

- $G = \mathbb{Z}_{p^k}, p$ prime (equivariant K-theory).


- $G = (\mathbb{Z}_p)^k$ be the $p$-torus of rank $k$, $p$ a prime, or $G = T^k = (S^1)^k$ be a $k$-dimensional torus (Borel cohomology, Borel localization theorem, Borel cohomology of stable cohomotopy theory).

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$$d(W) < d(V)/p^{k-1},$$

then $\phi(V, W) \geq 0$ (e.g., it implies that there is no $G$-equivariant map from $S(V)$ into $S(W)$).
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Remark
Note that opposite to the case $G = \mathbb{Z}_p$, for $G = \mathbb{Z}_{p^k}$ the classical formulation of the Borsuk-Ulam theorem does not hold.

In Bartsch Lecture Notes [3] (Th. 3.22) it is shown that for any (finite) group $G$ which contains an element $g$ of order $p^2$ there exist two complex representations $V, W$, $\dim V > \dim W$, and an equivariant map $f : S(V) \rightarrow S(W)$.

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Let \( A \) be a set of \( G \)-spaces (usually, it is a family of orbits). Let \( X \) be a \( G \)-spaces.

**Definition**

The \((A, K^*_G)\) – cup length of \( X \) is the **smallest** \( r \) such that there exist \( A_1, A_2, \ldots, A_r \in A \) and \( G \)-maps 

\[ \beta_i : A_i \to X, \ 1 \leq i \leq r, \]

with the following property:

\( \forall \gamma \in K^*_G(X) \) and \( \forall \omega_i \in \ker \beta_i^* \),

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We can consider the notion of $(A, K^*_G, I) – length index$ defined in a little bit different way.

For the equivariant cohomology theory $K^*_G$ and a $G$-space $X$, the cohomology

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is a $R$-module over the coefficient ring $R =: K^*_G(pt)$ via the natural $G$-map $p_X : X \to pt$.

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$$\omega \cdot \gamma = p^*_X(\omega) \cup \gamma, \text{ and } \omega_1 \cdot \omega_2 = \omega_1 \cup \omega_2,$$

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$\forall \gamma \in K_G^*(X)$ and $\forall \omega_i \in I \cap \ker(R \to K_G^*(A_i)) = \ker(K_G(\text{pt}) \to K_G(A_i))$, $i = 1, 2, \ldots, r$, the product

$$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_r \cdot \gamma = 0 \in K_G^*(X).$$
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Given two powers $1 \leq m \leq n \leq p^{k-1}$ of $p$, we put

$$A_{m,n} := \{ G/H | H \subset G; \, m \leq |H| \leq n \}, \quad (1)$$

where $|H|$ is the cardinality of $H$.

Next we put

$$l_n(X) = (A_{m,n}, K^*_G, I) - \text{length index of } (X). \quad (2)$$

The index $l_n$ does not depend on $m$.

Taking $A' = \{ G/H | |H| = n \}$ we can show that

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Therefore, to compute this index it is enough to consider

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The following result of Bartsch ([2]) is fundamental for the estimate from below of the index of $Z_f$.

We denote by $A_X$ a set of all $G$-orbits of $X$.

Theorem A.
Let $V$ be an orthogonal representation of $G = \mathbb{Z}_p^k$ with $V^G = \{0\}$ and $d = d(V) = \frac{1}{2} \dim_R V$. Fix $m$, $n$ two powers of $p$ as above. Then

$$
\ln(S(V)) \geq \begin{cases} 
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\infty & \text{if } A_{1m} \not\subset A_{n1}
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RELATION BETWEEN THE length index AND DIMENSION

How to relate the length index in $K_G(X)$ and the covering dimension of $X$?

G. Segal showed that there exists an Atiyah-Hirzebruch spectral sequence for the equivariant K-theory

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Consider the filtration of $K^*_G(X)$ associated to this spectral sequence: we

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Lemma B. If the subgroups of \( G \) are totally ordered and \( H \) is the largest isotropic subgroup on \( X \), then

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Theorem B. Let \( X \) be a compact \( G \)-space, with \( G = \mathbb{Z}_{p^k} \), and suppose that \( A_X \subset A_{m,n} \). If \( l_n(X) \geq r + 1 \) then \( \dim X = \dim X/G \geq 2r. \)
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Theorem 1. Let $V, W$ be two complex orthogonal representations of $G = \mathbb{Z}_{p^k}, p$ prime, such that $V^G = W^G = \{0\}$. Consider $f : S(V) \xrightarrow{G} W$ an equivariant map. Suppose $A_{S(V)} \subset A_{m,n}$ and $A_{S(W)} \subset A_{m,n}$. Then

$$l_n(Z_f) \geq 1 + \left\lfloor \frac{(d(V) - 1)m}{n} \right\rfloor - d(W).$$

Consequently,

$$\dim(Z_f) \geq 2 \left( \left\lfloor \frac{(d(V) - 1)m}{n} \right\rfloor - d(W) \right) := \phi(V, W).$$

In particular, if $d(W) < d(V)/p^{k-1}$, then $\phi(V, W) \geq 0$, which means that there is no $G$-equivariant map from $S(V)$ into $S(W)$. 
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Idea of the proof.

- Using monotonicity and additivity properties of the length index we show that

\[ l_n(Z_f) \geq l_n(S(V)) - l_n(S(W)). \]

- By assumption, \( A_{S(V)} \subset A_{m,n} \) and from Theorem A

\[ l_n(Z_f) \geq 1 + \left\lfloor \frac{(d(V) - 1)m}{n} \right\rfloor - d(W). \]

- Consequently, from Theorem B,

\[ \dim(Z_f) \geq 2 \left( \left\lfloor \frac{(d(V) - 1)m}{n} \right\rfloor - d(W) \right) := \phi(V, W). \]
Idea of the proof.

- Using monotonicity and additivity properties of the length index we show that

\[ l_n(Z_f) \geq l_n(S(V)) - l_n(S(W)). \]

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Theorem 2.
Let $V, W$ be two real orthogonal representations of $G = \mathbb{Z}_{2^k}$, with $V^G = W^G = \{0\}$. Consider $f : S(V) \xrightarrow{G} W$ an equivariant map. Suppose that $A_{S(V)} \subset A_{m,n}$ and $A_{S(W)} \subset A_{m,n}$.
Then
\[
\dim(Z_f) \geq \left[\frac{(d(V) - 1)m}{n}\right] - d(W) = \phi(V, W).
\]

In particular, if $d(W) < \frac{d(V)}{2^{k-1}}$, then $\phi(V, W) \geq 0$, which means that there is no $G$-equivariant map from $S(V)$ into $S(W)$. 
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Munkholm, H. J. *On the Borsuk-Ulam theorem for \( \mathbb{Z}_p^a \)-actions on \( S^{2n-1} \) and maps \( S^{2n-1} \rightarrow \mathbb{R}^m \).* Osaka J. Math. 7 (1970) 451-456.


