Pseudo-Anosov flows in graph manifolds with periodic pieces

Russ Waller

Florida State University
**Definition** Let $\Phi$ be a flow on a closed 3-manifold $M$. We say that $\Phi$ is a *pseudo-Anosov* flow if the following conditions are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x, t)$ is $C^1$, it is not a single point, and the tangent vector bundle $D_t\Phi$ is $C^0$ in $M$.

- There are two (possibly) singular transverse foliations $\Lambda^s, \Lambda^u$ which are two dimensional, with leaves saturated by the flow and so that $\Lambda^s, \Lambda^u$ intersect exactly along the flow lines of $\Phi$. 
• There are a finite number (possibly zero) of periodic orbits \(\{\gamma_i\}\), called singular orbits. A stable/unstable leaf containing a singularity is homeomorphic to \(P \times I/f\) where \(P\) is a \(p\)-prong in the plane and \(f\) is a homeomorphism from \(P \times \{1\}\) to \(P \times \{0\}\). In addition, \(p\) is at least 3.

• In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backward asymptotic.

**Definition** A pseudo-Anosov flow without singular orbits is an Anosov flow.
Manifolds that admit pseudo-Anosov flows
- have $\mathbb{R}^3$ as a universal cover
- have infinite fundamental group with exponential growth
**Definition** A *graph manifold* is an irreducible 3-manifold where all of the pieces of the torus decomposition are Seifert.

**Definition** In relation to a pseudo-Anosov flow, a Seifert fibered piece is *periodic* if the piece admits a Seifert fibration for which a regular fiber is freely homotopic to a closed orbit of the flow.

**Definition** A graph manifold in which all pieces of the torus decomposition are periodic is *totally periodic*. 
Fundamental objective: Find new examples of totally periodic graph manifolds with pseudo-Anosov flow.

Method:
- Show that totally periodic graph manifolds with pseudo-Anosov flow can be described using what are called fat graphs.
- Study fat graphs.
Definition A *Birkhoff annulus* is an immersed annulus so that each boundary component is a closed orbit of the flow and the interior of the annulus is transverse to the flow.
Let $I = [-\pi/2, \pi/2]$. Let $N = I \times S^1 \times I$ with coordinates $(x, y, z)$, thinking of $S^1$ as $[0, 1]/0 \sim 1$. For every real number $\lambda > 0$, define the $C^\infty$ vector field $X_\lambda$ defined by

\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= \lambda \sin(x) \cos^2(z) \\
\dot{z} &= \cos^2(x) + \sin^2(z) \sin^2(x)
\end{align*}
\]

The Birkhoff annulus associated to the block $N$ is

\[
B = [-\pi/2, \pi/2] \times S^1 \times \{0\}.
\]
**Definition** Given a surface $\Sigma$ with boundary that retracts onto a graph $X$, $\Sigma$ is a *fat graph* for $X$ and $X$ is *flow graph* if:

(i) the valence of every vertex is an even number.

(ii) the set of boundary components of $\Sigma$ can be partitioned into two subsets so that for every edge $e$ of $X$, the two sides of $e$ in $\Sigma$ lie in different subsets of this partition.

**Note** We do not require $\Sigma$ to be orientable.

**Remark** A vertex of valence $2p$ corresponds to a $p$-prong.
**Definition** A flow graph is *irreducible* if each vertex has a valence of at least 4.

**Definition** An irreducible flow graph is a *pseudo-Anosov graph* if each of the boundary components of the corresponding surface retract onto an even number of edges when the surface is retracted onto the graph.

**Definition** An *Anosov graph* is a 4-regular pseudo-Anosov graph.
Theorem 1 (W) Spheres with 2, 3, or 5 boundary components do not admit pseudo-Anosov graphs.
Theorem 2 (W) Orientable surfaces of genus $g$ with $b$ boundary components and $x \leq b - x$ incoming boundary components admit a pseudo-Anosov graph with $v$ vertices if and only if

- $b \geq 2$,
- $v + b$ is even,
- $x \geq 1 - g + (b - v)/2$, and
- $v \leq b - 2 + 2g$, with strict inequality if $v$ is odd and $g = 0$. 


Theorem 3 (W) The connected sum of $k$ real projective planes with $b$ boundary components and $x \leq b - x$ incoming boundary components admits a pseudo-Anosov graph with $v$ vertices if and only if

- $b \geq 2$,
- $v + b + k$ is even,
- $x \geq 1 + (b - v - k)/2$, and
- $v \leq b - 2 + k$. 
Corollary 4 (W) \( \text{Punctured spheres do not admit Anosov graphs with an odd number of vertices.} \)
Corollary 5 (W) Orientable surfaces of positive genus admit an Anosov graph if and only if

- $b \geq 2$,
- $v + b$ is even,
- $x \geq 1 - g + (b - v)/2$, and
- $v = b - 2 + 2g$. 
Corollary 6 (W) The connected sum of $k$ real projective planes admits an Anosov graph if and only if

- $b \geq 2$,
- $v + b + k$ is even,
- $x \geq 1 + (b - v - k)/2$, and
- $v = b - 2 + k$.
Theorem 7 (W) A $b$-punctured sphere that admits a pseudo-Anosov graph with $v$ vertices admits a pseudo-Anosov graph whose vertices have valence $\alpha_1, \ldots, \alpha_v$ if and only if

1. $\alpha_1 + \ldots + \alpha_v = 2v + 2b - 4$, and
2. some subset of $\{\alpha_1, \ldots, \alpha_v\}$ sums to $(\alpha_1 + \ldots + \alpha_v)/2$. 
Theorem 8 (W) Any orientable surface of positive genus and any non-orientable surface that admits a pseudo-Anosov graph with $v$ vertices admits a pseudo-Anosov graph whose vertices have valence $\alpha_1, ..., \alpha_v$ if and only if

\[ \alpha_1 + ... + \alpha_v = 2v + 2b + 4g - 4 \quad \text{or} \quad \alpha_1 + ... + \alpha_v = 2v + 2b + 2k - 4, \text{ respectively.} \]
Thank you