Pseudo-Anosov flows in graph manifolds with periodic pieces

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Definition Let Φ be a flow on a closed 3-manifold M. We say that Φ is a *pseudo-Anosov* flow if the following conditions are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x,t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0 in M.
- There are two (possibly) singular transverse foliations Λ^s, Λ^u which are two dimensional, with leaves saturated by the flow and so that Λ^s, Λ^u intersect exactly along the flow lines of Φ.

- There are a finite number (possibly zero) of periodic orbits $\{\gamma_i\}$, called *singular* orbits. A stable/unstable leaf containing a singularity is homeomorphic to $P \times I/f$ where P is a p-prong in the plane and f is a homeomorphism from $P \times \{1\}$ to $P \times \{0\}$. In addition, p is at least 3.
- In a stable leaf all orbits are forward asymptotic, in an unstable leaf all orbits are backward asymptotic.

Definition A pseudo-Anosov flow without singular orbits is an *Anosov* flow.

Manifolds that admit pseudo-Anosov flows

- have $\mathbb{R}^3$ as a universal cover
- have infinite fundamental group with exponential growth

Definition A *graph manifold* is an irreducible 3-manifold where all of the pieces of the torus decomposition are Seifert.

Definition In relation to a pseudo-Anosov flow, a Seifert fibered piece is *periodic* if the piece admits a Seifert fibration for which a regular fiber is freely homotopic to a closed orbit of the flow.

Definition A graph manifold in which all pieces of the torus decomposition are periodic is *totally periodic*.

Fundamental objective: Find new examples of totally periodic graph manifolds with pseudo-Anosov flow.

Method:

- Show that totally periodic graph manifolds with pseudo-Anosov flow can be described using what are called *fat graphs*.
- Study fat graphs.

Definition A *Birkhoff annulus* is an immersed annulus so that each boundary component is a closed orbit of the flow and the interior of the annulus is transverse to the flow.

Let $I = [-\pi/2, \pi/2]$. Let $N = I \times \mathbb{S}^1 \times I$ with coordinates (x, y, z), thinking of $\mathbb{S}^1$ as $[0, 1]/0 \sim 1$. For every real number $\lambda > 0$, define the C^∞ vector field X_λ defined by

$$\dot{x} = 0$$
$$\dot{y} = \lambda \sin(x) \cos^2(z)$$
$$\dot{z} = \cos^2(x) + \sin^2(z) \sin^2(x)$$

The Birkhoff annulus associated to the block N is

$$B = [-\pi/2, \pi/2] \times \mathbb{S}^1 \times \{0\}.$$

Definition Given a surface Σ with boundary that retracts onto a graph X, Σ is a *fat graph* for X and X is *flow graph* if:

(i) the valence of every vertex is an even number.

(ii) the set of boundary components of Σ can be partitioned into two subsets so that for every edge e of X, the two sides of e in Σ lie in different subsets of this partition.

Note We do not require Σ to be orientable.

Remark A vertex of valence $2p$ corresponds to a p-prong.

Definition A flow graph is *irreducible* if each vertex has a valence of at least 4.

Definition An irreducible flow graph is a *pseudo-Anosov graph* if each of the boundary components of the corresponding surface retract onto an even number of edges when the surface is retracted onto the graph.

Definition An *Anosov graph* is a 4-regular pseudo-Anosov graph.

Theorem 1 (W) *Spheres with 2,3, or 5 boundary components do not admit pseudo-Anosov graphs.*

Theorem 2 (W) *Orientable surfaces of genus g with b boundary components and $x \leq b - x$ incoming boundary components admit a pseudo-Anosov graph with v vertices if and only if*

- $b \geq 2$,
- $v + b$ *is even,*
- $x \geq 1 - g + (b - v)/2$, *and*
- $v \leq b - 2 + 2g$, *with strict inequality if v is odd and* $g = 0$.

Theorem 3 (W) *The connected sum of k real projective planes with b boundary components and $x \leq b - x$ incoming boundary components admits a pseudo-Anosov graph with v vertices if and only if*

- $b \geq 2$,
- $v + b + k$ *is even,*
- $x \geq 1 + (b - v - k)/2$, *and*
- $v \leq b - 2 + k$.

Corollary 4 (W) *Punctured spheres do not admit Anosov graphs with an odd number of vertices.*

Corollary 5 (W) *Orientable surfaces of positive genus admit an Anosov graph if and only if*

- $b \geq 2$,
- $v + b$ *is even,*
- $x \geq 1 - g + (b - v)/2$, *and*
- $v = b - 2 + 2g$.

Corollary 6 (W) *The connected sum of k real projective planes admits an Anosov graph if and only if*

- $b \geq 2$,
- $v + b + k$ *is even,*
- $x \geq 1 + (b - v - k)/2$, *and*
- $v = b - 2 + k$.

Theorem 7 (W) *A b-punctured sphere that admits a pseudo-Anosov graph with v vertices admits a pseudo-Anosov graph whose vertices have valence $\alpha_1, ..., \alpha_v$ if and only if*

- $\alpha_1 + ... + \alpha_v = 2v + 2b - 4,$ *and*
- *some subset of* $\{\alpha_1, ..., \alpha_v\}$ *sums to* $(\alpha_1 + ... + \alpha_v)/2$.

Theorem 8 (W) *Any orientable surface of positive genus and any non-orientable surface that admits a pseudo-Anosov graph with v vertices admits a pseudo-Anosov graph whose vertices have valence $\alpha_1, ..., \alpha_v$ if and only if*

$\alpha_1 + ... + \alpha_v = 2v + 2b + 4g - 4$ *or*

$\alpha_1 + ... + \alpha_v = 2v + 2b + 2k - 4$*, respectively.*

Thank you